Chapter 11

Lie derivative

Given some diffeomorphism φ, we have Eq. (10.5) for pushforwards and pullbacks,

$$(\varphi_* v) f = \left(\varphi^{-1}\right)^* v\left(\varphi^*(f)\right) . \qquad (11.1)$$

We will apply this to the flow ϕ_t of a vector field u, defined by

$$\frac{d}{dt}\left(\phi_t^* f\right)\Big|_{t=0} = u(f)\Big|_Q . \qquad (11.2)$$

Applying this at $-t$, we get

$$\begin{aligned} \phi_{-t*} v(f) &= \left(\phi_{-t}^{-1}\right)^* v\left(\phi_{-t}^*(f)\right) \\ &= \phi_t^* v\left(\phi_{-t}^*(f)\right) , \end{aligned} \qquad (11.3)$$

where we have used the relation $\phi_t^{-1} = \phi_{-t}$. Let us differentiate this equation with t,

$$\frac{d}{dt}(\phi_{-t*} v)(f)\Big|_{t=0} = \frac{d}{dt}\phi_t^* v\left(\phi_{-t}^*(f)\right)\Big|_{t=0} \qquad (11.4)$$

On the right hand side, ϕ_t^* acts linearly on vectors and v acts linearly on functions, so we can imagine $A_t = \phi_t^* v$ as a kind of linear operator acting on the function $f_t = \left(\phi_{-t}^* f\right)$. Then the right hand side is of the form

$$\begin{aligned} \frac{d}{dt} A_t f_t\Big|_{t=0} &= \left(\frac{d}{dt} A_t\right) f_t\Big|_{t=0} + A_t \frac{d}{dt} f_t\Big|_{t=0} \\ &= \left(\frac{d}{dt}\phi_t^* v\right) f_t\Big|_{t=0} + A_t\left(\frac{d}{dt}\phi_{-t}^*(f)\right)\Big|_{t=0} \\ &= u\left(v(f)\right)\Big|_{t=0} - v\left(u(f)\right)\Big|_{t=0} \\ &= [u\, , v](f)\Big|_{t=0} . \end{aligned} \qquad (11.5)$$

The things in the numerator are numbers, so they can be compared at different points, unlike vectors which may be compared only on the same space. We can also write this as

$$\lim_{t\to 0} \frac{\phi_{t*} v_{\phi_t(P)} - v_P}{t} = [u\,,v]\,. \tag{11.6}$$

• This has the look of a derivative, and it can be shown to have the properties of a derivation on the module of vector fields, appropriately defined. So the Lie bracket is also called the **Lie derivative**, and written as

$$\pounds_u v = [u\,,v]\,. \tag{11.7}$$

The derivation on functions by a vector field $u : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})\,, f \mapsto u(f)$, can be defined similarly as

$$u(f) = \lim_{t\to 0} \frac{\phi_t^* f - f}{t}\,. \tag{11.8}$$

• So this can also be called the **Lie derivative** of f with respect to u, and written as $\pounds_u f$. □

Then it is easy to see that

$$\begin{aligned} \pounds_u(fg) &= (\pounds_u f)\,g + f\,(\pounds_u g)\,, \\ \text{and} \qquad \pounds_u(f + ag) &= \pounds_u f + a\pounds_u g\,. \end{aligned} \tag{11.9}$$

So $\pounds_u$ is a derivation on the space $C^\infty(\mathcal{M})$. Also,

$$\begin{aligned} \pounds_u(v + aw) &= \pounds_u v + a\pounds_u w\,, \\ \text{and} \qquad \pounds_u(fv) &= (\pounds_u f)\,v + f\pounds_u v \qquad \forall f \in C^\infty(\mathcal{M}) \end{aligned} \tag{11.10}$$

So $\pounds_u$ is a derivation on the module of vector fields. Also, using Jacobi identity, we see that

$$\pounds_u(v \bullet w) = (\pounds_u v) \bullet w + v \bullet (\pounds_u w)\,, \tag{11.11}$$

where $v \bullet w = [v\,,w]$, so $\pounds_u$ is a derivation on the Lie algebra of vector fields.

Lie derivatives are useful in physics because they describe invariances. For functions, $\pounds_u f = 0$ means $\phi_t^* f = f$, so the function does not change along the flow of u. So the flow of u preserves f, or leaves f invariant.

If there are two vector fields u and v which leave f invariant, $\pounds_u f = 0 = \pounds_v f$. But we know from the Eq. (11.8), which defines the Lie derivative of a function that

$$\begin{aligned} & \pounds_{u+av} f = \pounds_u f + a\pounds_v = 0 \qquad \forall a \in \mathbb{R} \\ \text{and} \qquad & [\pounds_u\,, \pounds_v] f = \pounds_{[u\,,v]} f = 0\,. \end{aligned} \tag{11.12}$$

So the vector fields which preserve f form a Lie algebra.

Similarly, a vector field is invariant under a diffeomorphism φ if $\varphi_* v = v$, as mentioned earlier. Using the flow of u, we find that a vector field v is invariant under the flow of u if

$$\begin{aligned} & \phi_{-t} v = v \\ \Rightarrow \qquad & \pounds_u v = v\,. \end{aligned} \tag{11.13}$$

So if a vector field w is invariant under the flows of u and v, i.e. if $\pounds_u w = 0 = \pounds_v w$, we find that

$$0 = \pounds_u \pounds_v w - \pounds_v \pounds_u w = \pounds_{[u\,,v]} w\,. \tag{11.14}$$

Thus again the vector fields leaving w invariant form a Lie algebra.

• Let us also define the corresponding operations for 1-forms. As we mentioned in Chap. 6, a **1−form** is a section of the **cotangent bundle**

$$T^*\mathcal{M} = \bigcup_P T^*_P \mathcal{M}\,. \tag{11.15}$$

Alternatively, a 1-form is a smooth linear map from the space of vector fields on $\mathcal{M}$ to the space of smooth functions on $\mathcal{M}$,

$$\omega : v \mapsto \omega(v) \in C^\infty(\mathcal{M}), \qquad \omega(u + av) = \omega(U) + a\omega(v)\,. \tag{11.16}$$

A 1-form is a rule that (smoothly) selects a cotangent vector at each point. □

• Given a smooth map $\varphi \mathcal{M}_1 \to \mathcal{M}_2$ (say a diffeomorphism, for convenience), the **pullback** $\varphi^*\omega$ is defined by

$$\left(\varphi^*\omega\right)(v) = \omega\left(\varphi_*\omega\right)\,. \tag{11.17}$$

• We have already seen the **gradient** 1-form for a function $f : \mathcal{M} \to \mathbb{R}$, which is a linear map from the space of vector fields to functions,

$$df(u + av) = u(f) + av(f)\,, \tag{11.18}$$

and which can be written as

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{11.19}$$

in some chart. □

For an arbitrary 1-form ω, we can write in a chart and for any vector field v,

$$\omega = \omega_i dx^i\,, \qquad v = v^i \frac{\partial}{\partial x^i}\,, \qquad \omega(v) = \omega_i v^i\,. \tag{11.20}$$

All the components $\omega_i\,, v^i$ are smooth functions, so is $\omega_I v^i$. The space of 1-forms is a module. Since the function $\omega(v)$ is chart-independent, we can find the components $\omega_{i'}$ of ω in a new chart by noting that

$$\omega(v) = \omega_i v^i = \omega'_i v^{i'}\,. \tag{11.21}$$

Note that the notation is somewhat ambiguous here – i' also runs from 1 to n, and the prime actually distinguished the chart, or the coordinate system, rather than the index i.

If the components of v in the new chart are related to those in the old one by $v^{i'} = A^{i'}_j v^j$, it follows that

$$\omega_{i'} A^{i'}_j v^j = \omega_j v^j \qquad \Rightarrow \qquad \omega_{i'} A^{i'}_j = \omega_j \tag{11.22}$$

Since coordinate transformations are invertible, we can multiply both sides of the last equation by A^{-1} and write

$$\omega_{i'} = \left(A^{-1}\right)^j_{i'} \omega_j\,. \tag{11.23}$$

For coordinate transformations from a chart $\{x^i\}$ to a chart $\{x'^{i'}\}$,

$$A^{i'}_j = \frac{\partial x'^{i'}}{\partial x^j}\,, \qquad \left(A^{-1}\right)^j_{i'} = \frac{\partial x^j}{\partial x'^{i'}} \tag{11.24}$$

$$\text{so} \qquad v^{i'} = \frac{\partial x'^{i'}}{\partial x^j} v^j\,, \qquad \omega_{i'} = \frac{\partial x^j}{\partial x'^{i'}} \omega_j\,. \tag{11.25}$$

We can define the Lie derivative of a 1-form very conveniently by going to a chart, and treating the components of 1-forms and vector fields as functions,

$$\begin{aligned} \pounds_u \omega(v) = \pounds_u \left(\omega_i v^i\right) &= u^j \frac{\partial}{\partial x^j} \left(\omega_i v^i\right) \\ &= u^j \frac{\partial \omega_i}{\partial x^j} v^i + u^j \omega_i \frac{\partial v^i}{\partial x^j}\,. \end{aligned} \tag{11.26}$$

But we want to define things such that

$$\pounds_u\omega(v) = (\pounds_u\omega)(v) + \omega(\pounds_u v)\,. \tag{11.27}$$

We already know the left hand side of this equation from Eq. (11.26), and the right hand side can be calculated in a chart as

$$\begin{aligned}(\pounds_u\omega)(v) + \omega(\pounds_u v) &= (\pounds_u\omega)_i\, v^i + \omega_i\,(\pounds_u v)^i \\ &= (\pounds_u\omega)_i\, v^i + \omega_i[u\,,v]^i \\ &= (\pounds_u\omega)_i\, v^i + \omega_i\left(u^j\frac{\partial v^i}{\partial x^j} - v^j\frac{\partial u^i}{\partial x^j}\right)\,.\end{aligned} \tag{11.28}$$

Equating the right hand side of this with the right hand side of Eq. (11.26), we can write

$$(\pounds_u\omega)_i = u^j\frac{\partial\omega_i}{\partial x^j} + \omega_j\frac{\partial u^j}{\partial x^i}\,. \tag{11.29}$$

These are the components of $\pounds_u\omega$ in a given chart $\{x^i\}$.

For the sake of convenience, let us write down the Lie derivatives of the coordinate basis vector fields and basis 1-forms. The coordinate basis vector corresponding to the i-th coordinate is

$$v = \frac{\partial}{\partial x^i} \qquad \Rightarrow \qquad v^j = \delta^j_i\,. \tag{11.30}$$

Putting this into the formula for Lie derivatives, we get

$$\begin{aligned}\pounds_u\frac{\partial}{\partial x^i} &= [u\,,v]^j\frac{\partial}{\partial x^j} \\ &= \left(u^k\frac{\partial v^j}{\partial x^k} - v^k\frac{\partial u^j}{\partial x^k}\right)\frac{\partial}{\partial x^j} \\ &= \left(0 - \delta^k_i\frac{\partial u^j}{\partial x^k}\right)\frac{\partial}{\partial x^j} \\ &= -\left(\frac{\partial u^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}\,.\end{aligned} \tag{11.31}$$

Similarly, the 1-form corresponding to the i-th basis coordinate is

$$dx^i = \delta^i_j dx^j\,, \qquad i.e. \qquad \left(dx^i\right)_j = \delta^i_j\,. \tag{11.32}$$

Using this in the formula Eq. (11.29) we get

$$\pounds_u dx^i = \delta^i_k\frac{\partial u^k}{\partial x^j}dx^j = \frac{\partial u^i}{\partial x^j}dx^j\,. \tag{11.33}$$

There is also a geometric description of the Lie derivative of 1-forms,

$$\begin{aligned} \pounds_u \omega\big|_P &= \lim_{t\to 0} \frac{1}{t}\left[\phi_t^* \omega\big|_{\phi_t(P)} - \omega_P\right] \\ &= \frac{d}{dt}\phi_t^*\omega\Big|_P \,. \end{aligned} \tag{11.34}$$

We will not discuss this in detail, but only mention that it leads to the same Leibniz rule as in Eq. (11.27), and the same description in terms of components as in Eq. (11.29).