# Chapter 12

# Tensors

So far, we have defined tangent vectors, cotangent vectors, and also vector fields and 1-forms. We will now define tensors. We will do this by starting with the example of a specific type of tensor.

• A (1, 2) **tensor** $A_P$ at $P \in \mathcal{M}$ is a map

$$A_P : T_P\mathcal{M} \times T_P\mathcal{M} \times T^*_P\mathcal{M} \to \mathbb{R} \tag{12.1}$$

which is linear in every argument. □

So given two vectors $u_P, v_P$ and a covector $\omega_P$ ,

$$A_P : (u_P, v_P, \omega_P) \mapsto A_P\,(u_P, v_P; \omega_P) \in \mathbb{R}\,. \tag{12.2}$$

Suppose $\{e_a\}, \{\lambda^a\}$ are bases for $T_P\mathcal{M}, T^*_P\mathcal{M}$ . Write

$$A^c_{ab} = A_P\,(e_a, e_b; \lambda^c)\ . \tag{12.3}$$

Then for arbitrary vectors $u_P = u^a e_a$ , $v_P = v^a e_a$ , and covector $\omega_P = \omega_a \lambda^a$ we get using linearity of the tensor map,

$$\begin{aligned} A_P\,(u_P, v_P; \omega_P) &= A_P\left(u^a e_a, v^b e_b; \omega_c \lambda^c\right) \\ &= u^a v^b \omega_c A^c_{ab}\,. \end{aligned} \tag{12.4}$$

It is a matter of convention whether $A$ as written above should be called a (1, 2) tensor or a (2, 1) tensor, and the convention varies between books. So it is best to specify the tensor by writing indices as there is no confusion about $A^c_{ab}$.

A tensor of type $(p, q)$ can be defined in the same way,

$$A^{p,q}_P : \underbrace{T_P\mathcal{M} \times \cdots \times T_P\mathcal{M}}_{q \text{ times}} \times \underbrace{T^*_P\mathcal{M} \times \cdots \times T^*_P\mathcal{M}}_{p \text{ times}} \quad \to \quad \mathbb{R} \tag{12.5}$$

in such a way that the map is linear in every argument.

• Alternatively, $A_P$ is an element of the **tensor product space**

$$A_P \in \underbrace{T_P\mathcal{M} \otimes \cdots \otimes T_P\mathcal{M}}_{p\,\text{times}} \otimes \underbrace{T_P^*\mathcal{M} \otimes \cdots \otimes T_P^*\mathcal{M}}_{q\,\text{times}} \tag{12.6}$$

We can define the components of this tensor in the same way that we did for the (1, 2) tensor. Then a $(p, q)$ tensor has components which can be written as

$$A^{a_1\cdots a_p}_{b_1\cdots b_q}\,.$$

• Some special types of $(p, q)$ tensors have special names. A (1, 0) tensor is a linear map $A_P : T^*_P\mathcal{M} \to \mathbb{R}$, so it is a tangent vector. A (0, 1) tensor is a cotangent vector. A $(p,\ 0)$ tensor has components with $p$ upper indices. It is called a **contravariant $p-$tensor**. A $(0,\ q)$ tensor has components with $q$ lower indices. It is called a **covariant $q-$tensor**. □

It is possible to add tensors of the same type, but not of different types,

$$A^{a_1\cdots a_p}_{b_1\cdots b_q} + B^{a_1\cdots a_p}_{b_1\cdots b_q} = (A+B)^{a_1\cdots a_p}_{b_1\cdots b_q}\,. \tag{12.7}$$

• A **tensor field** is a rule giving a tensor at each point. □

We can now define the Lie derivative of a tensor field by using Leibniz rule in a chart. Let us first consider the components of a tensor field in a chart. For a (1, 2) tensor field $A$, the components in a chart are

$$A^k_{ij} = A(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}; dx^k)\,. \tag{12.8}$$

The components are functions of $x$ in a chart. Thus we can write this tensor field as

$$A = A^k_{ij} dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}\,, \tag{12.9}$$

where the $\times$ indicates a 'product', in the sense that its action on two vectors and a 1-form is a product of the respective components,

$$\left(dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}\right)(u, v; \omega) = u^i v^j \omega_k\,. \tag{12.10}$$

Thus we find, in agreement with the earlier definition,

$$A(u, v; \omega) = A^k_{ij} u^i v^j \omega_k\,. \tag{12.11}$$

Under a change of charts, i.e. coordinate system $x^i \to x'^{i'}$, the components of the tensor field change according to

$$A = A^k_{ij}\, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} = A^{k'}_{i'j'}\, dx'^{i'} \otimes dx'^{j'} \otimes \frac{\partial}{\partial x'^{k'}} \tag{12.12}$$

Since

$$dx'^{i'} = \frac{\partial x'^{i'}}{\partial x^i} dx^i\,, \qquad \frac{\partial}{\partial x'^{i'}} = \frac{\partial x^i}{\partial x'^{i'}} \frac{\partial}{\partial x^i} \tag{12.13}$$

($i$ and $i'$ are not equal in general), we get

$$A^k_{ij}\, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} = A^{k'}_{i'j'}\, \frac{\partial x'^{i'}}{\partial x^i} dx^i \otimes \frac{\partial x'^{j'}}{\partial x^j} dx^j \otimes \frac{\partial x^k}{\partial x'^{k'}} \frac{\partial}{\partial x^k}\,. \tag{12.14}$$

Equating components, we can write

$$A^k_{ij} = A^{k'}_{i'j'}\, \frac{\partial x'^{i'}}{\partial x^i} \frac{\partial x'^{j'}}{\partial x^j} \frac{\partial x^k}{\partial x'^{k'}} \tag{12.15}$$

$$A^{k'}_{i'j'} = A^k_{ij}\, \frac{\partial x^i}{\partial x'^{i'}} \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x'^{k'}}{\partial x^k}\,. \tag{12.16}$$

From now on, we will use the notation $\partial_i$ for $\dfrac{\partial}{\partial x^i}$ and $\partial_i f$ for $\dfrac{\partial f}{\partial x^i}$ unless there is a possibility of confusion. This will save some space and make the formulae more readable.

We can calculate the Lie derivative of a tensor field (with respect to a vector field $u$, say) by using the fact that $\pounds_u$ is a derivative on the modules of vector fields and 1-forms, and by assuming Leibniz rule for tensor products. Consider a tensor field

$$T = T^{m\cdots n}_{a\cdots b}\, \partial_m \otimes \cdots \otimes \partial_n \otimes dx^a \otimes \cdots \otimes dx^b\,. \tag{12.17}$$

Then

$$\begin{aligned}\pounds_u T = &\left(\pounds_u T^{m\cdots n}_{a\cdots b}\right)\, \partial_m \otimes \cdots \otimes \partial_n \otimes dx^a \otimes \cdots \otimes dx^b \\ &+ T^{m\cdots n}_{a\cdots b}\, (\pounds_u \partial_m) \otimes \cdots \otimes \partial_n \otimes dx^a \otimes \cdots \otimes dx^b + \cdots \\ &+ T^{m\cdots n}_{a\cdots b}\, \partial_m \otimes \cdots \otimes \partial_n \otimes (\pounds_u dx^a) \otimes \cdots \otimes dx^b\,, + \cdots \end{aligned} \tag{12.18}$$

where the dots stand for the terms involving all the remaining upper and lower indices. Since the components of a tensor field are functions on the manifold, we have

$$\pounds_u T^{m\cdots n}_{a\cdots b} = u^i \partial_i T^{m\cdots n}_{a\cdots b}\,, \tag{12.19}$$

and we also know that

$$\pounds_u \partial_m = -\frac{\partial u^i}{\partial x^m}\partial_i \,, \qquad \pounds_u dx^a = \frac{\partial u^a}{\partial x^i} dx^i \,. \tag{12.20}$$

Putting these into the expression for the Lie derivative for $T$ and relabeling the dummy indices, we find the components of the Lie derivative,

$$\begin{aligned}(\pounds_u T)^{m\cdots n}_{a\cdots b} = \; & u^i\, \partial_i T^{m\cdots n}_{a\cdots b} \\ & - T^{i\cdots n}_{a\cdots b}\, \partial_i u^m - \cdots - T^{m\cdots i}_{a\cdots b}\, \partial_i u^n \\ & + T^{m\cdots n}_{i\cdots b}\, \partial_a u^i + \cdots + T^{m\cdots n}_{a\cdots i}\, \partial_b u^i \,. \end{aligned} \tag{12.21}$$