Chapter 17

Hodge duality

We will next define the Hodge star operator. We will defineit in a chart rather than abstractly.

• The **Hodge star operator**, denoted $\star$ in an n-dimensional manifold is a map from p-forms to $(n-p)$-forms given by

$$(\star\omega)_{\mu_1\cdots\mu_{n-p}} \equiv \frac{\sqrt{|g|}}{p!}\,\epsilon_{\mu_1\cdots\mu_n}\,g^{\mu_{n-p+1}\nu_1}\,\cdots g^{\mu_n\nu_p}\,\omega_{\nu_1\cdots\nu_p}\,, \tag{17.1}$$

where ω is a p-form. □

The $\star$ operator acts on forms, not on components.

Example: Consider $\mathbb{R}^3$ with metric $+++$, i.e. $g_{\mu\nu} = \text{diag}(1,1,1)$. Then $|g| \equiv g = 1\,, g^{\mu\nu}\text{diag}(1,1,1)$. Write the coordinate basis 1-forms as dx, dy, dz. Their components are clearly

$$(dx)_i = \delta_i^1\,,\ (dy)_i = \delta_i^2\,,\ (dz)_i = \delta_i^3\,, \tag{17.2}$$

the δ's on the right hand sides are Kroenecker deltas. So

$$\begin{aligned}
(\star dx)_{ij} &= \epsilon_{ijk}g^{kl}(dx)_l = \epsilon_{ijk}g^{kl}\delta_l^1 = \epsilon_{ijk}g^{k1} \\
\Rightarrow \qquad \star dx &= \frac{1}{2!}(\star dx)_{ij}dx^i\wedge dx^j = \frac{1}{2!}\epsilon_{ijk}g^{k1}dx^i\wedge dx^j \\
&\qquad g^{k1} = 1 \text{ for k} = 1\,, 0 \text{ otherwise} \\
\Rightarrow \qquad \star dx &= \frac{1}{2!}\left(dx^2\wedge dx^3 - dx^3\wedge dx^2\right) = dx^2\wedge dx^3 = dy\wedge dz\,.
\end{aligned} \tag{17.3}$$

Similarly, $\star dy = dz\wedge dx\,, \qquad \star dz = dx\wedge dy\,.$ □

Example: Consider $p = 0$ (scalar), i.e. a 0-form ω in n dimensions.

$$\begin{aligned} (\star\omega)_{\mu_1\cdots\mu_n} &= \sqrt{|g|}\epsilon_{\mu_1\cdots\mu_n}\omega \\ \Rightarrow \qquad (\star 1)_{\mu_1\cdots\mu_n} &= \sqrt{|g|}\epsilon_{\mu_1\cdots\mu_n} \\ \Rightarrow \qquad (\star 1) &= \frac{\sqrt{|g|}}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} \\ &= dV \end{aligned} \tag{17.4}$$

□

Example: $p = n$. Then

$$(\star\omega) = \frac{\sqrt{|g|}}{n!}\epsilon_{\mu_1\cdots\mu_n} g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\omega_{\nu_1\cdots\nu_n} . \tag{17.5}$$

For the volume form,

$$\begin{aligned} dV &= \frac{\sqrt{|g|}}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} \\ (dV)_{\nu_1\cdots\nu_n} &= \sqrt{|g|}\epsilon_{\nu_1\cdots\nu_n} \\ (\star dV) &= \frac{|g|}{n!}\epsilon_{\mu_1\cdots\mu_n} g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\epsilon_{\nu_1\cdots\nu_n} \\ &= \frac{|g|}{n!}n!(\det g)^{-1} = \frac{|g|}{n!}\frac{n!}{g} = sign(g) = (-1)^s , \end{aligned} \tag{17.6}$$

where s is the number of (-1) in $g_{\mu\nu}$. □

So we find that

$$\star(\star 1) = \star dV = (-1)^s , \tag{17.7}$$

and

$$\star(\star dV) = (-1)^s(\star 1) = (-1)^s dV , \tag{17.8}$$

i.e., $(\star)^2 = (-1)^s$ on 0-forms and n-forms.

In general, on a p-form in an n-dimensional manifold with signature $(s, n-s)$, it can be shown in the same way that

$$(\star)^2 = (-1)^{p(n-p)+s} . \tag{17.9}$$

In particular, in four dimensional Minkowski space, $s = 1, n = 4$, so

$$(\star)^2 = (-1)^{p(4-p)+1} . \tag{17.10}$$

It is useful to work out the Hodge dual of basis p-forms. Suppose we have a basis p-form $dx^{I_1} \wedge \cdots \wedge dx^{I_p}$, where the indices are arranged in increasing order $I_p > \cdots > I_1$. Then its components are $p!\delta^{I_1}_{\mu_1} \cdots \delta^{I_p}_{\mu_p}$. So

$$\begin{aligned}
&\star \left(dx^{I_1} \wedge \cdots \wedge dx^{I_p}\right)_{\nu_1 \cdots \nu_{n-p}} \\
&\qquad = \frac{\sqrt{|g|}}{p!}\, \epsilon_{\nu_1 \cdots \nu_{n-p}\mu_1 \cdots \mu_p} g^{\mu_1 \mu'_1} \cdots g^{\mu_p \mu'_p}\, p!\, \delta^{I_1}_{\mu'_1} \cdots \delta^{I_p}_{\mu'_p} \\
&\qquad = \sqrt{|g|}\, \epsilon_{\nu_1 \cdots \nu_{n-p}\mu_1 \cdots \mu_p} g^{\mu_1 I_1} \cdots g^{\mu_p I_p} \,. \qquad (17.11)
\end{aligned}$$

We will use this to calculate $\star\omega \wedge \omega$.

For a p-form ω, we have

$$\begin{aligned}
\omega &= \frac{1}{p!}\omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\
&= \sum_I \omega_{I_1 \cdots I_p} dx^{I_1} \wedge \cdots \wedge dx^{I_p} \qquad (17.12)
\end{aligned}$$

where the sum over I means a sum over all possible index sets $I = I_1 < \cdots < I_p$, but there is no sum over the indices $\{I_1, \cdots, I_p\}$ themselves, in a given index set the I_k are fixed. Using the dual of basis $p-$forms, and Eq. (13.13), we get

$$\begin{aligned}
\star\omega &= \sum_I \omega_{I_1 \cdots I_p} \star \left(dx^{I_1} \wedge \cdots \wedge dx^{I_p}\right) \\
&= \sum_I \frac{\sqrt{|g|}}{(n-p)!} \epsilon_{\nu_1 \cdots \nu_{n-p}\mu_1 \cdots \mu_p} g^{\mu_1 I_1} \cdots g^{\mu_p I_p}\, \omega_{I_1 \cdots I_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \,.
\end{aligned} \qquad (17.13)$$

The sum over I is a sum over different index sets as before, and the Greek indices are summed over as usual. Thus we calculate

$$\begin{aligned}
\star\omega \wedge \omega &= \frac{\sqrt{|g|}}{(n-p)!} \sum_{I,J} \epsilon_{\nu_1 \cdots \nu_{n-p}\mu_1 \cdots \mu_p} g^{\mu_1 I_1} \cdots g^{\mu_p I_p} \omega_{I_1 \cdots I_p} \times \\
&\qquad dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \wedge \left(\omega_{J_1 \cdots J_p} dx^{J_1} \wedge \cdots \wedge dx^{J_p}\right) \\
&= \frac{\sqrt{|g|}}{(n-p)!} \sum_{I,J} \epsilon_{\nu_1 \cdots \nu_{n-p}\mu_1 \cdots \mu_p} g^{\mu_1 I_1} \cdots g^{\mu_p I_p} \omega_{I_1 \cdots I_p} \omega_{J_1 \cdots J_p} \times \\
&\qquad dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \wedge dx^{J_1} \wedge \cdots \wedge dx^{J_p} \qquad (17.14)
\end{aligned}$$

We see that the set $\{\nu_1, \cdots, \nu_{n-p}\}$ cannot have any overlap with the set $J = \{J_1, \cdots, J_p\}$, because of the wedge product. On the other hand, $\{\nu_1, \cdots, \nu_{n-p}\}$ cannot have any overlap with $\{\mu_1, \cdots, \mu_p\}$ because ϵ is totally antisymmetric in its indices. So the set $\{\mu_1, \cdots, \mu_p\}$ must have the same elements as the set $J = \{J_1, \cdots, J_p\}$, but they may not be in the same order.

Now consider the case where the basis is orthogonal, i.e. $g^{\mu\nu}$ is diagonal. Then $g^{\mu_k I_k} = g^{I_k I_k}$ etc. and we can write

$$\star\omega \wedge \omega = \frac{\sqrt{|g|}}{(n-p)!} \sum_{I,J} \epsilon_{\nu_1 \cdots \nu_{n-p} I_1 \cdots I_p} g^{I_1 I_1} \cdots g^{I_p I_p} \omega_{I_1 \cdots I_p} \omega_{J_1 \cdots J_p} \times dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \wedge dx^{J_1} \wedge \cdots \wedge dx^{J_p} . \quad (17.15)$$

We see that in each term of the sum, the indices $\{I_1 \cdots I_p\}$ must be the same as $\{J_1 \cdots J_p\}$ because both sets are totally antisymmetrized with the indices $\{\nu_1 \cdots \nu_{n-p}\}$.

Since both sets are ordered, it follows that we can replace J by I,

$$\begin{aligned} \star\omega \wedge \omega &= \frac{\sqrt{|g|}}{(n-p)!} \sum_{I} \epsilon_{\nu_1 \cdots \nu_{n-p} I_1 \cdots I_p} g^{I_1 I_1} \cdots g^{I_p I_p} \omega_{I_1 \cdots I_p} \omega_{I_1 \cdots I_p} \times \\ &\qquad dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \wedge dx^{I_1} \wedge \cdots \wedge dx^{I_p} \\ &= \frac{\sqrt{|g|}}{(n-p)!} \sum_{I} \epsilon_{\nu_1 \cdots \nu_{n-p} I_1 \cdots I_p} \omega^{I_1 \cdots I_p} \omega_{I_1 \cdots I_p} \times \\ &\qquad dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}} \wedge dx^{I_1} \wedge \cdots \wedge dx^{I_p} . \end{aligned} \quad (17.16)$$

In each term of this sum, the indices $\{\nu_1 \cdots \nu_{n-p}\}$ are completely determined, so we can replace them by the corresponding ordered set $K = K_1 < \cdots < K_{n-p}$, which is completely determined by the set I, so that

$$\star\omega \wedge \omega = \sqrt{|g|} \sum_{I} \epsilon_{K_1 \cdots K_{n-p} I_1 \cdots I_p} \omega^{I_1 \cdots I_p} \omega_{I_1 \cdots I_p} \times dx^{K_1} \wedge \cdots \wedge dx^{K_{n-p}} \wedge dx^{I_1} \wedge \cdots \wedge dx^{I_p} \quad (17.17)$$

The indices on this ϵ are a permutation of $\{1, \cdots, n\}$, so ϵ is ± 1. But this sign is the same as that for the permutation to bring the basis to the order $dx^1 \wedge \cdots \wedge dx^n$, so the overall sign to get both to the standard order is positive. Thus we get

$$\begin{aligned} \star\omega \wedge \omega &= \sqrt{|g|} \sum_I \omega^{I_1\cdots I_p} \omega_{I_1\cdots I_p} \epsilon_{1\cdots n}\, dx^1 \wedge \cdots \wedge dx^n \\ &= \sqrt{|g|}\, \frac{1}{p!} \omega^{\mu_1\cdots\mu_p} \omega_{\mu_1\cdots\mu_p}\, dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{p!} \omega^{\mu_1\cdots\mu_p} \omega_{\mu_1\cdots\mu_p}\, (vol) \end{aligned} \tag{17.18}$$

If we are in a basis where the metric is not diagonal, it is still symmetric. So we can diagonalize it locally by going to an appropriate basis, or set of coordinates, at each point. In this basis, the components of ω may be $\omega'_{\mu_1\cdots\mu_p}$, so we can write

$$\star\omega \wedge \omega = \left(\frac{1}{p!} \omega^{\mu'_1\cdots\mu'_p} \omega_{\mu'_1\cdots\mu'_p} \right) (vol') \tag{17.19}$$

But both factors are invariant under a change of basis. So we can now change back to our earlier basis, and find Eq. (17.18) even when the metric is not diagonal. Note that the metric may not be diagonalizable globally or even in an extended region.