Chapter 19

Stokes' theorem

We will next discuss a very beautiful result called Stokes' formula. This is actually a theorem, but we will not prove it, only state the result and discuss its applications. So for us it is only a formula, but still deep and beautiful.

• A **submanifold** $\mathcal{S}$ is a subset of points in $\mathcal{M}$ such that any point in $\mathcal{S}$ has an open neighbourhoood in $\mathcal{M}$ for which there is some chart where $(n-m)$ coordinates vanish. $\mathcal{S}$ is then m-dimensional. □

• Suppose $\mathcal{U}$ is a region of an oriented manifold $\mathcal{M}$. The **boundary** $\partial\mathcal{U}$ of $\mathcal{U}$ is a submanifold of dimension $n-1$ which divides $\mathcal{M}$ in such a way that any curve joining a point in $\mathcal{U}$ with a point in $\mathcal{U}^c$ must contain a point in $\partial\mathcal{U}$.

Now suppose $\mathcal{U}$ has an oriented smooth boundary $\partial\mathcal{U}$. Then $\partial\mathcal{U}$ is automatically an oriented manifold, by considering the restrictions of the charts on $\mathcal{U}$ to $\partial\mathcal{U}$.

• Consider a smooth $(n-1)$ form in $\mathcal{M}$. **Stokes' formula** says that

$$\int\limits_{\mathcal{U}} d\omega = \int\limits_{\partial\mathcal{U}} \omega\,. \tag{19.1}$$

If $\mathcal{M}$ is a compact manifold with boundary $\partial\mathcal{M}$, this formula can be applied to all of $\mathcal{M}$. If ω vanishes outside some compact region we can again set $\mathcal{U} = \mathcal{M}$. Also, $\mathcal{U}$ can be a submanifold in another manifold, like a 2-surface in a 3-manifold. □

Example: Let $\mathcal{U} = [0,1]$. Then a function $f : \mathcal{M} \to \mathbb{R}$ is a 0-form, and $df = f'(x)dx$ is a 1-form. Take the orientation of $\mathcal{M}$ to be from 0 to 1. Then $\partial\mathcal{M}$ consists of the points $x = 0$ and $x = 1$, and Stokes' formula says that

$$\int_{\mathcal{M}} df = \int_{\partial\mathcal{M}} f$$

$$i.e. \qquad \int_0^1 f'(x)dx = f(1) - f(0)\,. \tag{19.2}$$

Example: Consider a 2-d disk D in $\mathbb{R}^2$, with boundary ∂D. Take a 1-form A. Then Stokes' formula says

$$\int_{\partial D} A = \int_D dA\,. \tag{19.3}$$

Let us seee this equation in a chart. We can write

$$\begin{aligned} A &= A_i\, dx^i \\ dA &= \partial_i A_j\, dx^i \wedge dx^j \end{aligned} \tag{19.4}$$

A evaluated on ∂D can be written as $A\left(\frac{d}{dt}\right)$ where $\frac{d}{dt}$ is tangent to ∂D. So we can write $A\left(\frac{d}{dt}\right) = A_i \frac{dx^i}{dt} dt$, and

$$\begin{aligned} \int_{\partial D} A_i dx^i &= \int_D \partial_i A_j\, dx^i \wedge dx^j \\ &= \int_D \left(\partial_1 A_2 - \partial_2 A_1\right)\, dx^1 \wedge dx^2 \\ &= \int_{\varphi(D)} \left(\partial_1 A_2 - \partial_2 A_1\right)\, d^2x\,. \end{aligned} \tag{19.5}$$

Similarly for higher forms on higher dimensional manifolds.

- ***Gauss' divergence theorem*** is a special case of Stokes' theorem. Before getting to Gauss' theorem, we need to make a new definition. Consider an n-form $\omega \neq 0$ on an n-dimensional manifold. We can write this in a chart as

$$\begin{aligned} \omega &= f dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{n!} f \epsilon_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}\,. \end{aligned} \tag{19.6}$$

Given a vector field v, its contraction with ω is

$$\begin{aligned}\iota_v\omega = \omega(v,\cdots) &= \frac{1}{(n-1)!}\omega_{\mu_1\mu_2\cdots\mu_n}\, v^{\mu_1} dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_n} \\ &= fv^1 dx^2\wedge\cdots\wedge dx^n - fv^2 dx^1\wedge dx^3\wedge\cdots\wedge dx^n + \cdots\end{aligned} \tag{19.7}$$

Then we can calculate

$$\begin{aligned}d(\iota_v\omega) = d\omega(v,\cdots) &= \partial_1(fv^1)\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n \\ &\quad +\partial_2(fv^2)\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n \\ &\quad +\cdots+\partial_n(fv^n)\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n \\ &= \partial_\mu(fv^\mu)\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n \\ &= \frac{1}{f}\partial_\mu(fv^\mu)\,\omega\,.\end{aligned} \tag{19.8}$$

In particular, if ω is the volume form, we can write

$$\begin{aligned}\omega &= \frac{\sqrt{|g|}}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}\,, \\ d(\iota_v(vol)) &= \frac{1}{\sqrt{|g|}}\partial_\mu(v^\mu\sqrt{|g|})(vol)\,.\end{aligned} \tag{19.9}$$

- This is called the ***divergence*** of the vector field v.

There is another expression for the divergence. Remember that given a vector field v, we can define a one-form, also called v, with components defined with the help of the metric,

$$v_\mu = g_{\mu\mu'}v^{\mu'} \tag{19.10}$$

Consider $\star v$, which has components

$$\begin{aligned}(\star v)_{\mu_1\cdots\mu_{n-1}} &= \sqrt{|g|}\epsilon_{\mu_1\cdots\mu_{n-1}\mu}g^{\mu\mu'}v_{\mu'} \\ &= \sqrt{|g|}\epsilon_{\mu_1\cdots\mu_{n-1}\mu}v^\mu\,.\end{aligned} \tag{19.11}$$

$$\begin{aligned}\Rightarrow\qquad \star v &= \frac{\sqrt{|g|}}{(n-1)!}\epsilon_{\mu_1\cdots\mu_{n-1}\mu}v^\mu dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n-1}} \\ d\star v &= \partial_{\mu_n}\left(\frac{\sqrt{|g|}}{(n-1)!}\epsilon_{\mu_1\cdots\mu_{n-1}\mu}v^\mu\right) dx^{\mu_n}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n-1}} \\ &= \frac{(-1)^{n-1}}{(n-1)!}\epsilon_{\mu_1\cdots\mu_{n-1}\mu}\left(\partial_{\mu_n}\left(\sqrt{|g|}v^\mu\right)\right) dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}\end{aligned} \tag{19.12}$$

Both μ and μ_n must be different from $(\mu_1, \cdots, \mu_{n-1})$, so $\mu = \mu_n$. Thus in each term of the sum, the choice of $(\mu_1, \cdots, \mu_{n-1})$ automatically selects $\mu_n (= \mu)$, so a sum over $(\mu_1, \cdots, \mu_n)$ overcounts n times. So we can write

$$\begin{aligned} d\star v &= \frac{(-1)^{n-1}}{(n-1)!}\epsilon_{\mu_1\cdots\mu_n}\left(\partial_\mu\left(\sqrt{|g|}v^\mu\right)\right)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} \\ &= (-1)^{n-1}\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}v^\mu)(vol)\,. \end{aligned} \tag{19.13}$$

Since this is an n-form in n dimensions, we can calculate from here that

$$\star d\star v = \frac{(-1)^{n+s-1}}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}v^\mu)\,, \tag{19.14}$$

where as before s is the signature of the manifold, i.e. the number of negative entries in the metric in a locally diagonal form.

Let us now go back to Stokes' formula. Take a region $\mathcal{U}$ of $\mathcal{M}$ which is covered by a single chart and has an orientable boundary $\partial\mathcal{U}$ as before. Then we find

$$\begin{aligned} \int\limits_{\mathcal{U}} \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}v^\mu)(vol) &= \int\limits_{\mathcal{U}} d(\iota_v(vol)) \\ &= \int\limits_{\partial\mathcal{U}} \iota_v(vol)\,. \end{aligned} \tag{19.15}$$

Now suppose b is a 1-form normal to $\partial\mathcal{U}$, i.e. $b\left(\frac{d}{dt}\right) = 0$ for any vector $\frac{d}{dt}$ tangent to $\partial\mathcal{U}$, and α is an $(n-1)$-form such that $b \wedge \alpha = (vol)$. Since all n-forms are proportional, α always exists. For the same reason, if $b \neq 0$ on $\partial\mathcal{U}$, it is unique up to a factor. And $b \neq 0$ on $\partial\mathcal{U}$ because $\partial\mathcal{U}$ is defined as the submanifold where one coordinate is constant, usually set to zero, so that one component of $\frac{d}{dt}$ vanishes at any point on $\partial\mathcal{U}$, and therefore the corresponding component of b can be chosen to be non-zero.

So b is unique up to a rescaling $b \to b' = fb$ for some nonvanishing function f. But we can always scale $\alpha \to \alpha' = f^{-1}\alpha$ so that $b' \wedge \alpha' = b \wedge \alpha$. Further, if we restrict α to $\partial\mathcal{U}$, i.e. if α acts only on tangent vectors to $\partial\mathcal{U}$, we find that α is an $(n-1)$-form on an $(n-1)$-dimensional manifold, so it is unique up to scaling. Therefore, α is unique once b is given. Finally, for any vector v,

$$\iota_v(vol)\Big|_{\partial\mathcal{U}} = \iota_v(b\wedge\alpha)\Big|_{\partial\mathcal{U}} \tag{19.16}$$

is an $(n-1)$-form on $\partial\mathcal{U}$ which acts only on vectors tangent to $\partial\mathcal{U}$. Then

$$\iota_v(b\wedge\alpha)\Big|_{\partial\mathcal{U}} = b(v)\alpha\Big|_{\partial\mathcal{U}} \tag{19.17}$$

because all terms of the form $b\wedge\iota_v\alpha$ gives zero for any choice of $(n-1)$ vectors on $\partial\mathcal{U}$.

Then we have

$$\begin{aligned}\int\limits_{\mathcal{U}} \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}v^\mu)(vol) &= \int\limits_{\partial\mathcal{U}} b(v)\alpha \\ &= \int\limits_{\partial\mathcal{U}} (n_\mu v^\mu)\,\alpha\,. \end{aligned} \tag{19.18}$$

Usually b is taken to have norm 1. Then α is the volume form on $\partial\mathcal{U}$, and we can write

$$\int\limits_{\mathcal{U}} \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}v^\mu)(vol) = \int\limits_{\partial\mathcal{U}} (n_\mu v^\mu)\sqrt{|g_{(\partial\mathcal{U})}|}d^{n-1}x\,. \tag{19.19}$$