Chapter 2

Manifolds

Now that we have the notions of open sets and continuity, we are ready to define the fundamental object that will hold our attention during this course.

- A **manifold** is a topological space which is locally like $\mathbb{R}^n$. □

That is, every point of a manifold has an open neighbourhood with a one-to-one map onto some open set of $\mathbb{R}^n$.

- More precisely, a topological space M is a **smooth n-dimensional manifold** if the following are true:

i) We can **cover** the space with open sets U_α, i.e. every point of M lies within some U_α.

ii) $\exists$ a map $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, where φ_α is one-to-one and onto some open set of $\mathbb{R}^n$. φ_α is continuous, ${\varphi_\alpha}^{-1}$ is continuous, i.e. $\varphi_\alpha \to V_\alpha \in \mathbb{R}^n$ is a homeomorphism for V_α.

$(U_\alpha, \varphi_\alpha)$ is called a **chart** (U_α is called the **domain** of the chart). The collection of charts is called an **atlas**.

iii) In any intersection $U_\alpha \cap U_\beta$, the maps $\varphi_\alpha \circ {\varphi_\beta}^{-1}$, which are called **transition functions** and take open sets of $\mathbb{R}^n$ to open sets of $\mathbb{R}^n$, i.e. $\varphi_\alpha \circ {\varphi_\beta}^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$, are smooth maps. □

- n is called the **dimension** of M. □

We have defined smooth manifolds. A more general definition is that of a C^k manifold, in which the transition functions are C^k, i.e. k times differentiable. Smooth means k is large enough for the purpose at hand. In practice, k is taken to be as large as necessary, up to C^∞. We get **real analytic manifolds** when the transition functions are real analytic, i.e. have a Taylor expansion at each point, which converges. Smoothness of a manifold is useful because then we can say unambiguously if a function on the manifold is smooth as we will see below.

• A **complex analytic manifold** is defined similarly by replacing $\mathbb{R}^n$ with $\mathbb{C}^n$ and assuming the transitions functions $\varphi_\alpha \circ {\varphi_\beta}^{-1}$ to be holomorphic (complex analytic). □

• Given a chart $(U_\alpha, \varphi_\alpha)$ for a neighbourhood of some point P, the image $(x_1, \cdots, x_n) \in \mathbb{R}^n$ of P is called the **coordinates** of P in the chart $(U_\alpha, \varphi_\alpha)$. A chart is also called a **local coordinate system**.□

In this language, a manifold is a space on which a local coordinate system can be defined, and coordinate transformations between different local coordinate systems are smooth. Often we will suppress U and write only φ for a chart around some point in a manifold. We will always mean a smooth manifold when we mention a manifold.

Examples: $\mathbb{R}^n$ (with the usual topology) is a manifold. □

The typical example of a manifold is the sphere. Consider the sphere S^n as a subset of $\mathbb{R}^{n+1}$:

$$(x^1)^2 + \cdots + (x^{n+1})^2 = 1 \tag{2.1}$$

It is not possible to cover the sphere by a single chart, but it is possible to do so by two charts.[1]

For the two charts, we will construct what is called the **stereographic projection**. It is most convenient to draw this for a circle in the plane, i.e. S^1 in $\mathbb{R}^2$, for which the equatorial 'plane' is simply an infinite straight line. Of course the construction works for any S^n. consider the 'equatorial plane' defined as the $x^1 = 0$, i.e. the set $\{(0, x^2, \cdots, x^{n+1})\}$, which is simply $\mathbb{R}^n$ when we ignore the first zero. We will find homeomorphisms from open sets on S^n to open sets on this $\mathbb{R}^n$. Let us start with the north pole $\mathcal{N}$, defined as the point $(1, 0, \cdots, 0)$.

We draw a straight line from $\mathcal{N}$ to any point on the sphere. If that point is in the upper hemisphere $(x^1 > 0)$ the line is extended till it hits the equatorial plane. The point where it hits the plane is the

[1] The reason that it is not possible to cover the sphere with a single chart is that the sphere is a compact space, and the image of a compact space under a continuous map is compact. Since $\mathbb{R}^n$ is non-compact, there cannot be a homeomorphism between S^n and $\mathbb{R}^n$.

image of the point on the sphere which the line has passed through. For points on the lower hemisphere, the line first passes through the equatorial plane (image point) before reaching the sphere (source point). Then using similarity of triangles we find (Exercise!) that the coordinates on the equatorial plane $\mathbb{R}^n$ of the image of a point on $S^n \backslash \{\mathcal{N}\}$ is given by

$$\varphi_N : \left(x^1, x^2, \cdots, x^{n+1}\right) \mapsto \left(\frac{x^2}{1-x^1}, \cdots, \frac{x^{n+1}}{1-x^1}\right). \tag{2.2}$$

Similarly, the stereographic projection from the south pole is

$$\begin{aligned} \varphi_S &: S^n \backslash \{\mathcal{S}\} \to \mathbb{R}^n, \\ &\left(x^1, x^2, \cdots, x^{n+1}\right) \mapsto \left(\frac{x^2}{1+x^1}, \cdots, \frac{x^{n+1}}{1+x^1}\right). \end{aligned} \tag{2.3}$$

If we write

$$z = \left(\frac{x^2}{1-x^1}, \cdots, \frac{x^{n+1}}{1-x^1}\right), \tag{2.4}$$

we find that

$$|z|^2 \equiv \left(\frac{x^2}{1-x^1}\right)^2 + \cdots + \left(\frac{x^{n+1}}{1-x^1}\right)^2 = \frac{1-(x^1)^2}{(1-x^1)^2} = \frac{1+x^1}{1-x^1} \tag{2.5}$$

The overlap between the two sets is the sphere without the poles. Then the transition function between the two projections is

$$\varphi_S \circ \varphi_N : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}, \qquad z \mapsto \frac{z}{|z|^2}. \tag{2.6}$$

These are differentiable functions of z in $\mathbb{R}^n \backslash \{0\}$. This shows that the sphere is an n-dimensional differentiable manifold. □

• A **Lie group** is a group G which is also a smooth (real analytic for the cases we will consider) manifold such that group composition written as a map $(x, y) \mapsto xy^{-1}$ is smooth. □

Another way of defining a Lie group is to start with an n-**parameter continuous group** G which is a group that can be parametrized by n (and only n) real continuous variables. n is called the **dimension** of the group, $n = \dim G$. (This is a different definition of the dimension. The parameters are global, but do not in general form a global coordinate system.)

Then any element of the group can be written as $g(a)$ where $a = (a_1, \cdots, a_n)$. Since the composition of two elements of G must be another element of G, we can write $g(a)g(b) = g(\phi(a,b))$ where $\phi = (\phi_1, \cdots, \phi_n)$ are n functions of a and b. Then for a **Lie group**, the functions ϕ are smooth (real analytic) functions of a and b.

These definitions of a Lie group are equivalent, i.e. define the same objects, if we are talking about finite dimensional Lie groups. Further, it is sufficient to define them as smooth manifolds if we are interested only in finite dimensions, because all such groups are also real analytic manifolds. Apparently there is another definition of a Lie group as a **topological group** (like n-parameter continuous group, but without an a priori restriction on n, in which the composition map $(x, y) \mapsto xy^{-1}$ is continuous) in which it is always possible to find an open neighbourhood of the identity which does not contain a subgroup.

Any of these definitions makes a Lie group a smooth manifold, an n-dimensional Lie group is an n-dimensional manifold. □

The phase space of N particles is a $6N$-dimensional manifold, $3N$ coordinates and $3N$ momenta. □

The Möbius strip is a 2-dimensional manifold. □

The space of functions with some specified properties is often a manifold. For example, linear combinations of solutions of Schrödinger equation which vanish outside some region form a manifold. □

Finite dimensional vector spaces are manifolds. □

Infinite dimensional vector spaces with finite norm (e.g. Hilbert spaces) are manifolds. □

• A **connected** manifold cannot be written as the disjoint union of open sets. Alternatively, the only subsets of a connected manifold which are both open and closed are $\emptyset$ and the manifold itself. □

SO(3), the group of rotations in three dimensions, is a 3-dimensional connected manifold. O(3), the group of rotations plus reflections in three dimensions, is also a 3-dimensional manifold, but is not connected since it can be written as the disjoint union SO(3)∪PSO(3) where P is reflection. □

$\mathcal{L}_+^\uparrow$, the group of proper (no space reflection) orthochronous (no time reflection) Lorentz transformations, is a 6-dimensional connected manifold. The full Lorentz group is a 6-dimensional manifold, not connected. □

Rotations in three dimensions can be represented by 3×3 real **orthogonal** matrices R satisfying $R^{\mathsf{T}}R = \mathbb{I}$. Reflection is represented by the matrix $P = -\mathbb{I}$. The space of 3×3 real orthogonal matrices is a connected manifold. □

The space of all $n\times$ real non-singular matrices is called $\mathrm{GL}(n, R)$. This is an n^2-dimensional Lie group and connected manifold. □