Chapter 20

Lie groups

We start a brief discussion on Lie groups, mainly with an eye to their structure as manifolds and also application to the theory of fiber bundles.

• A **Lie group** is a group which is also an analytic manifold. □

We did not define a Lie group in this way in Chap. 2 , but said that a Lie group was a manifold in which the group product is analytic in the group parameters, or alternatively the group product and group inverse are both C^∞ .

The definition above comes from a theorem that given a continuous group G in which the group product and group inverse are C^∞ functions of the group parameters, it is always possible to find a set of coordinate charts covering G such that the overlap functions are real analytic, i.e. are C^∞ and their Taylor series at any point converge to their respective values.

• A **Lie subgroup** of G is a subset H of G which is a subgroup of G , a submanifold of G , and is a topological group, i.e., a topological space in which the group product and group inverse are continuous maps. □

• Sometimes this expressed in terms of another definition. $\mathcal{P}$ is an **immersed submanifold** of $\mathcal{M}$ if the inclusion map $j : \mathcal{P} \hookrightarrow \mathcal{M}$ is smooth and at each point $p \in \mathcal{P}$ its differential dj_p is one to one, with dj_p being defined by $dj_p : T_p\mathcal{P} \to T_{j(p)}\mathcal{M}$ such that $dj_p(v)(g) = v(g \cdot j_p)$. □

We have mentioned some specific examples of Lie groups earlier. Let us mention some more examples.

Example: $\mathbb{R}^n$ is a Lie group under addition. So is $\mathbb{C}^n$.

Example: $\mathbb{R}^n\backslash\{0\}$ is a Lie group under multiplication. So is $\mathbb{C}^n\backslash\{0\}$.

Example: The **direct product** of two Lie groups is itself a Lie group, with multiplication $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$. □

Example: The set of all $n \times n$ real invertible matrices forms a group under matrix multiplication, called the **General Linear group** $GL(n, \mathbb{R})$. This is also the space of all invertible linear maps of $\mathbb{R}^n$ to itself. We can similarly define $GL(n, \mathbb{C})$. □

The next few examples are Lie subgroups of $GL(n, \mathbb{R})$.

Example: The **Special Linear group** $SL(n, \mathbb{R})$ is the subset of $GL(n, \mathbb{R})$ for which all the matrices have determinant $+1$, i.e., $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | \det A = 1\}$. One can define $SL(n, \mathbb{C})$ in a similar manner. □

Example: The **Orthogonal group** $O(n) = \{R \in GL(n, \mathbb{R}) \mid R^{\mathsf{T}}R = \mathbb{I}\}$. □

Example: The **Unitary group** $U(n) = \{U \in GL(n, \mathbb{C}) \mid U^\dagger U = \mathbb{I}\}$. □

Example: The **Symplectic group** $Sp(n)$, defined as the subgroup of $U(2n)$ given by $A^{\mathsf{T}}\mathbb{J}A = \mathbb{J}$, where

$$\mathbb{J} = \begin{pmatrix} 0 & -\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

□

Example: $O(p, q) = \{R \in GL(p+q, \mathbb{R}) \mid R^{\mathsf{T}}\eta_{p,q}R = \eta_{p,q}\}$, where

$$\eta_{p,q} = \begin{pmatrix} \mathbb{1}_{p\times p} & 0 \\ 0 & -\mathbb{1}_{q\times q} \end{pmatrix}$$

□

Example: $U(p, q) = \{U \in GL(p + q, \mathbb{C}) \mid U^\dagger\eta_{p,q}U = \eta_{p,q}\}$.

Example The **Special Orthogonal group** $SO(n)$ is the subgroup of $O(n)$ for which determinant is $+1$. Similarly, the **Special unitary group** $SU(n)$ is the subgroup of $U(n)$ with determinant $+1$. Similarly for $SO(p.q)$ and $SU(p, q)$.

The group $U(1)$ is the group of phases $U(1) = \{e^{i\phi}|\phi \in \mathbb{R}\}$. As a manifold, this is isomorphic to a circle S^1.

The group $SU(2)$ is isomorphic as a manifold to a three-sphere S^3. These are the only two spheres (other than the point S^0) which admit a Lie group structure.

An important property of a Lie group is that the tangent space of any point is isomorphic to the tangent space at the identity by an appropriate group operation. Of course, the tangent space at any pooint of a manifold is isomorphic to the tangent space at any other point. For Lie groups, the isomorphism between the tangent spaces is induced by group operations, so is in some sense natural.

For any Lie group G, we can define diffeomorphisms of G labelled by elements $g \in G$, called

- **Left translation** $l_g : G \to G \qquad g' \mapsto gg'$; □
- **Right translation** $r_g : G \to G \qquad g' \mapsto g'g$. □

These can be defined for any group, but are diffeomorphisms for Lie groups. We see that

$$\begin{aligned} l_{g^{-1}} l_g(g') = l_{g^{-1}}(gg') = g^{-1}gg' = g' \quad &\Rightarrow \quad (l_g)^{-1} = l_{g^{-1}} \\ r_{g^{-1}} r_g(g') = r_{g^{-1}}(g'g) = g'gg^{-1} = g' \quad &\Rightarrow \quad (r_g)^{-1} = r_{g^{-1}} \,. \end{aligned} \tag{20.1}$$

It is easy to check that

$$l_{g_1} l_{g_2} = l_{g_1 g_2} \qquad r_{g_1} r_{g_2} = r_{g_2 g_1} \tag{20.2}$$

Further, $l_{g^{-1}}(g) = e$ and $r_{g^{-1}}(g) = e$, so any element of G can be moved to the identity by a diffeomorphism. The tangent space at the identity forms a Lie algebra, as we shall see. The left and right translations lead to diffeomorphisms which relate the tangent space at any point to this Lie algebra, as we shall see now.