Chapter 22

One parameter subgroups

There is another characterization of T_eG for a Lie group G as the set of its one parameter subgroups, which we will now define. This is also called the "infinitesimal" description of a Lie group, and what Lie called an infinitesimal group.

• A ***one parameter subgroup*** of a Lie group G is a smooth homomorphism from the additive group of real numbers to $G\,, \gamma : (\mathbb{R}, +) \to G$. Then $\gamma : \mathbb{R} \to G$ is a curve such that $\gamma(s+t) = \gamma(s)\gamma(t)\,, \gamma(0) = e$, and $\gamma(-t) = \gamma(t)^{-1}$. □

Also, since this is a homomorphism, the one parameter subgroup is Abelian.

Example: For $G = (\mathbb{R}\backslash\{0\}, \times)$ the multiplicative group of non-zero real numbers, $\gamma(t) = e^t$ is a 1-p subgroup.

Example: $G = U(1)\,, \qquad \gamma(t) = e^{it}\,.$

Example: $G = SU(2)\,, \qquad \gamma(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\,.$

Example: $G = GL(3, \mathbb{R})\,, \qquad \gamma(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}\,.$

The relation between 1-p subgroups and and T_eG is given by the

Theorem: The map $\gamma \mapsto \dot\gamma(0) = \dot\gamma\big|_e$ defines a 1-1 correspondence between 1-p subgroups of G, and $T_eG\,.$

Proof: For any $X \in T_eG$ define $L^X = l_{g*}X$ as the corresponding left-invariant vector field. We need to find a smooth homomorophism from $\mathbb{R}$ to G using L^X. This homomorphism is provided by the flow or integral curve of L^X, but let us work this out in more detail.

Denote the integral curve of L^X by $\gamma^X(t)$, i.e.,

$$\begin{aligned} \gamma^X(0) &= L^X_e = X \\ \text{and} \qquad \dot{\gamma}^X(t) &= L^X_{\gamma(t)} = l_{\gamma(t)*}X \end{aligned} \tag{22.1}$$

Since L^X is left-invariant, $l_{g'*}L^X_g = L^X_{g'g}$. Consider the equation

$$\frac{d}{dt}\gamma(t) = L^X_{\gamma(t)} = l_{\gamma(t)*}X \equiv \gamma_*\left(\frac{d}{dt}\right)\Big|_t \,. \tag{22.2}$$

Given some τ, replace $\gamma(t)$ by $\gamma(\tau + t)$ to get

$$\gamma(\tau + t) = l_{\gamma(\tau+t)*}X \,. \tag{22.3}$$

Remember that $\gamma(t)$ is an element of the group for each t. Now replace $\gamma(t)$ in Eq. (22.2) by $\gamma(\tau)\gamma(t)$ to get

$$\big(\gamma(\tau)\gamma(t)\big)_*\left(\frac{d}{dt}\right) = L^X_{\gamma(\tau)\gamma(t)} \tag{22.4}$$

We see that $\gamma(t+\tau)$ and $\gamma(\tau)\gamma(t)$ are both integral curves of L^X, i.e. both satisfy the equation of the integral curve of L^X, and at $t = 0$ both curves are at the point $\gamma(\tau)$. Thus by uniqueness these two are the same curve,

$$\gamma(\tau + t) = \gamma(\tau)\gamma(t)\,, \tag{22.5}$$

and $t \mapsto \gamma(t)$ is the homomorphism $\mathbb{R} \to G$ that we are looking for.

Thus for each $X \in T_eG$ we find a 1-p subgroup $\gamma(t)$ given by the integral curve of L^X,

$$\dot{\gamma}(t) = L^X_{\gamma(t)} = l_{\gamma(t)*}X\,, \tag{22.6}$$

where, as mentioned earlier, $(\dot{\gamma}(0) = X)$. □

In a compact connected Lie group G, every element lies on some 1-p subgroup. This is not true in a non-compact G, i.e. there are elements in G which do not lie on a 1-p subgroup. However, an Abelian non-compact group will always have a 1-p subgroup, so this remark applies only to non-Abelian non-compact groups.

For matrix groups, every 1-p subgroup is of the form

$$\gamma(t) = \left\{ e^{tM} \,\middle|\, M \text{ fixed, } t \in \mathbb{R} \right\}. \tag{22.7}$$

Let us see why. Suppose $\{\gamma(t)\}$ is a 1-p subgroup of the matrix group. Then $\gamma(t)$ is a matrix for each t, and

$$\gamma(s)\gamma(t) = \gamma(s+t)\,. \tag{22.8}$$

Differentiate with respect to s and set $s = 0$. Then

$$\dot{\gamma}(0)\gamma(t) = \dot{\gamma}(t)\,. \tag{22.9}$$

Write $\dot{\gamma}(0) = M$. Since G is a matrix group, M is a matrix. Then the unique solution for γ is

$$\gamma(t) = e^{tM}\,. \tag{22.10}$$

The properties of M are determined by the properties of the group and vice versa, not every matrix M will generate any group.

The allowed matrices $\{M\}$ for a given group G are the $\{\dot{\gamma}(0)\}$ for all the 1-p subgroups $\gamma(t)$, so these are in fact the tangent vectors at the identity. The allowed matrices $\{M\}$ for a given matrix group G thus form a Lie algebra with the Lie bracket being given by the matrix commutator. This Lie algebra is isomorphic to the Lie algebra of the group G. (We will not a give a proof of this here.)

We can find the Lie algebra of a matrix group by considering elements of the form $\gamma(t) = e^{tM}$ for small t, i.e.,

$$\gamma(t) = \mathbb{I} + tM \tag{22.11}$$

for small t. Conversely, once we are given, or have found, a Lie algebra with basis $\{t_i\}$, we can exponentiate the Lie algebra to find the set of 1-p subgroups

$$\left\{\gamma(a) = \exp a^i t_i\right\} \tag{22.12}$$

• This is the **infinitesimal group**, for compact connected groups this is identical to the Lie group itself. So in such cases, the entire group can be generated by exponentiating the Lie algebra. Non-compact groups cannot be written as the exponential of the Lie algebra in general. □

Example: Consider $SO(N)$, the group of $N\times N$ real orthogonal matrices R with $R^{\mathsf{T}}R = \mathbb{I}\,, \det R = 1$. Write $R = \mathbb{I} + A$, then $A^{\mathsf{T}} = -A$, i.e. the Lie algebra is spanned by $N \times N$ real antisymmetric matrices. Let us construct a basis for this algebra.

An $N \times N$ antisymmetric matrix has $N(N-1)/2$ independent elements. So we define $N(N-1)/2$ independent antisymmetric matrices, labelled by $\mu, \nu = 1, \cdots, N$,

$$\begin{aligned} M_{\mu\nu} &= -M_{\nu\mu} \qquad \mu\,, \nu \text{ are not matrix indices} \\ (M_{\mu\nu})_{\rho\sigma} &= (M_{\mu\nu})_{\sigma\rho}\,, \qquad \rho\,, \sigma \text{ are matrix indices}\,. \end{aligned} \tag{22.13}$$

A convenient choice for the basis is given by

$$(M_{\mu\nu})_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\,. \tag{22.14}$$

Then the commutators are calculated to be

$$[M_{\mu\nu}\,, M_{\alpha\beta}] = \delta_{\nu\alpha}M_{\mu\beta} - \delta_{\mu\alpha}M_{\nu\beta} + \delta_{\mu\beta}M_{\nu\alpha} - \delta_{\nu\beta}M_{\mu\alpha}\,. \tag{22.15}$$

This defines the Lie algebra.

Example: For $SU(N)$, the group of $N \times N$ unitary matrices U with $U^\dagger U = \mathbb{I}$, $\det U = 1$, the 1-p subgroups are given by $\gamma(t) = e^{tM}$ with $M^\dagger + M = 0$ in the same way as above, and $\det(\mathbb{I} + tM) = 1 \Rightarrow \operatorname{Tr} M = 0$. So the $SU(N)$ Lie algebra consists of traceless antihermitian matrices. Often the basis is multiplied by i to write $\gamma(a) = \exp(ia_j t_j)$, where t_j are now Hermitian matrices, with

$$[t_i\,, t_j] = if_{abc}t_c\,. \tag{22.16}$$