Chapter 3

Tangent vectors

Vectors on a manifold are to be thought in terms tangents to the manifold, which is a generalization of tangents to curves and surfaces, and will be defined shortly. But a tangent to a curve is like the velocity of a particle at that point, which of course comes from motion along the curve, which is its trajectory. And motion means comparing things at nearby points along the trajectory. And comparing functions at nearby points leads to differentiation. So in order to get to vectors, let us first start with the definitions of these things.

• A function $f : \mathcal{M} \to \mathbb{R}$ is **differentiable** at a point $P \in M$ if in a chart φ at P, the function $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\varphi(P)$. □

This definition does not depend on the chart. If $f \circ {\varphi_\alpha}^{-1}$ is differentiable at $\varphi_\alpha(P)$ in a chart $(U_\alpha, \varphi_\alpha)$ at P, the $f \circ {\varphi_\beta}^{-1}$ is differentiable at $\varphi_\beta(P)$ for any chart (U_β, φ_β) because

$$f \circ {\varphi_\beta}^{-1} = (f \circ {\varphi_\alpha}^{-1}) \circ (\varphi_\alpha \circ {\varphi_\beta}^{-1}) \tag{3.1}$$

and the transition functions $(\varphi_\alpha \circ {\varphi_\beta}^{-1})$ are differentiable.

This should be thought of as a special case of functions from one manifold to another. Consider two manifolds $\mathcal{M}$ and $\mathcal{N}$ of dimension m and n, and a mapping $f : \mathcal{M} \to \mathcal{N}$, $P \mapsto Q$. Consider local charts (U, φ) around P and (W, ψ) around Q. Then $\psi \circ f \circ \varphi^{-1}$ is a map from $\mathbb{R}^m \to \mathbb{R}^n$ and represents f in these local charts.

• f is **differentiable** at P if $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(P)$. In other words, f is differentiable at P if the coordinates $y^i = f^i(x^\mu)$ of Q are differentiable functions of the coordinates x^μ of P. □

• If f is a bijection (i.e. one-to-one and onto) and f and f^{-1} are both differentiable, we say that f is a **diffeomorphism** and that $\mathcal{M}$ and $\mathcal{N}$ are **diffeomorphic**. □

In all of these definitions, differentiable can be replaced by C^k or smooth.

• Two Lie groups are **isomorphic** if there is a diffeomorphism between them which is also a group homomorphism. □

• A **curve** in a manifold $\mathcal{M}$ is a map γ of a closed interval $\mathbb{R}$ to $\mathcal{M}$. (This definition can be given also when $\mathcal{M}$ is a topological space.) □

We will take this interval to be $I = [0,1] \subset \mathbb{R}$. Then a curve is a map $\gamma : I \to \mathcal{M}$. If $\gamma(0) = P$ and $\gamma(1) = P'$, for some γ, we say that γ joins P and P'.

• A manifold $\mathcal{M}$ is **connected** (actually **arcwise connected**)[1] if any two points in it can be joined by a continuous curve in $\mathcal{M}$. □

As for any map, a curve γ is called smooth iff its image in a chart is smooth in $\mathbb{R}^n$, i.e., iff $\varphi \circ \gamma : I \to \mathbb{R}^n$ is smooth in $\mathbb{R}^n$.

Note that the definition of a curve implies that it is parametrized. So the same collection of points in $\mathcal{M}$ can stand for two different curves if they have different parametrizations.

We are now ready to define tangent vectors and the tangent space to a manifold. There are different ways of defining tangent vectors.

i) Coordinate approach: Vectors are defined to be objects satisfying certain transformation rules under a change of chart, i.e. coordinate transformation, $(U_\alpha, \varphi_\alpha) \to (U_\beta, \varphi_\beta)$.

ii) Derivation approach: A vector is defined as a derivation of functions on the manifold. This is thinking of a vector as defining a "directional derivative".

iii) Curves approach: A vector tangent to a manifold is tangent to a curve on the manifold.

The approaches are equivalent in the sense that they end up defining the same objects and the same space. We will follow the third approach, or perhaps a mix of the second and the third approaches. Later we will briefly look at the derivation approach more carefully and compare it with the way we have defined tangent vectors.

Consider a smooth function $f : \mathcal{M} \to \mathbb{R}$. Given a curve $\gamma : I \to \mathcal{M}$, the map $f \circ \gamma : I \to \mathbb{R}$ is well-defined, with a well-defined

[1] It can be shown that an arcwise connected space is connected.

derivative. The rate of change of f along γ is written as $\dfrac{df}{dt}$.

Suppose another curve another curve $\mu(s)$ meets $\gamma(t)$ at some point P, where $s = s_0$ and $t = t_0$, such that

$$\frac{d}{dt}(f \circ \gamma)\Big|_P = \frac{d}{ds}(f \circ \mu)\Big|_P \qquad \forall f \in C^\infty(\mathcal{M}) \tag{3.2}$$

That is, we are considering a situation where two curves are tangent to each other in geometric and parametric sense. Let us introduce a convenient notation. In any chart φ containing the point P, let us write $\varphi(P) = (x^1, \cdots, x^n)$. Let us write $f \circ \gamma = (f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)$, so that the maps are

$$f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}, \qquad \boldsymbol{x} \mapsto f(\boldsymbol{x}) \text{ or } f(x^i) \tag{3.3}$$

$$\varphi \circ \gamma : I \to \mathbb{R}^n, \qquad t \mapsto \{x^i(\gamma(t))\}. \tag{3.4}$$

The last are the coordinates of the curve in $\mathbb{R}^n$.

Using the chain rule for differentiation, we find

$$\frac{d}{dt}(f \circ \gamma) = \frac{d}{dt} f(\boldsymbol{x}(\gamma(t))) = \frac{\partial f}{\partial x^i}\frac{dx^i(\gamma(t))}{dt}. \tag{3.5}$$

Similarly, for the curve μ we find

$$\frac{d}{ds}(f \circ \mu) = \frac{d}{ds} f(\boldsymbol{x}(\mu(s))) = \frac{\partial f}{\partial x^i}\frac{dx^i(\mu(s))}{ds}. \tag{3.6}$$

Since f is arbitrary, we can say that two curves γ, μ have the same tangent vector at the point $P \in \mathcal{M}$ (where $t = t_0$ and $s = s_0$) iff

$$\frac{dx^i(\gamma(t))}{dt}\bigg|_{t=t_0} = \frac{dx^i(\mu(s))}{ds}\bigg|_{s=s_0}. \tag{3.7}$$

We can say that these numbers completely determine the rate of change of any function along the curve γ or μ at P. So we can define the tangent to the curve.

- The **tangent vector** to a curve γ at a point P on it is defined as the map

$$\dot\gamma_P : C^\infty(\mathcal{M}) \to \mathbb{R}, \qquad f \mapsto \dot\gamma_P(f) \equiv \frac{d}{dt}\,(f \circ \gamma)\big|_P\,. \tag{3.8}$$

As we have already seen, in a chart with coordinates $\{x^i\}$ we can write using chain rule

$$\dot\gamma_P(f) = \frac{dx^i(\gamma(t))}{dt}\,\frac{\partial f}{\partial x^i}\bigg|_{\varphi(P)} \tag{3.9}$$

The numbers $\left.\dfrac{dx^i(\gamma(t))}{dt}\right|_{\varphi(P)}$ are thus the components of $\dot\gamma_P$. We will often write a tangent vector at P as v_P without referring to the curve it is tangent to.

We note here that there is another description of tangent vectors based on curves. Let us write $\gamma \sim \mu$ if γ and μ are tangent to each other at the point P. It is easy to see, using Eq. (3.7) for example, that this relation $\sim$ is transitive, reflexive, and symmetric. In other words, $\sim$ is an equivalence relation, for which the equivalence class $[\gamma]$ contains all curves tangent to γ (as well as to one another) at P.

• A **tangent vector** at $P \in \mathcal{M}$ is an equivalence class of curves under the above equivalence relation. □

The earlier definition is related to this by saying that if a vector v_P is tangent to some curve γ at P, i.e. if $v_P = \dot\gamma_P$, we can write $v_P = [\gamma]$.