Chapter 4

Tangent Space

• The set of all tangent vectors (to all curves) at some point $P \in \mathcal{M}$ is the ***tangent space*** $T_P\mathcal{M}$ at P. □

Proposition: $T_P\mathcal{M}$ is a vector space with the same dimensionality n as the manifold $\mathcal{M}$.

Proof: We need to show that $T_P\mathcal{M}$ is a vector space, i.e.

$$X_P + Y_P \in T_P\mathcal{M}\,, \tag{4.1}$$

$$aX_P \in T_P\mathcal{M}\,, \tag{4.2}$$

$$\forall X_P, Y_P \in T_P\mathcal{M}, \qquad a \in \mathbb{R}.$$

That is, given curves γ, μ passing through P such that $X_P = \dot{\gamma}_P, Y_P = \dot{\mu}_P$, we need a curve λ passing through P such that $\dot{\lambda}_P(f) = X_P(f) + Y_P(f) \forall f \in C^\infty(\mathcal{M})$.

Define $\overline{\lambda} : I \to \mathbb{R}^n$ in some chart φ around P by $\overline{\lambda} = \varphi \circ \gamma + \varphi \circ \mu - \varphi(P)$. Then $\overline{\lambda}$ is a curve in $\mathbb{R}^n$, and

$$\lambda = \varphi^{-1} \circ \overline{\lambda} \, : \, I \to M \tag{4.3}$$

is a curve with the desired property. □

Note: we cannot define $\lambda = \gamma + \mu - P$ because addition does not make sense on the right hand side.

The proof of the other part works similarly. (Exercise!)

To see that $T_P\mathcal{M}$ has n basis vectors, we consider a chart φ with coordinates x^i. Then take n curves λ_k such that

$$\varphi \circ \lambda_k(t) = \Big(x^1(P), \cdots, x^k(P) + t, \cdots, x^n(P)\Big)\,, \tag{4.4}$$

i.e., only the k-th coordinate varies along t. So λ_k is like the axis of the k-th coordinate (but only in some open neighbourhood of P).

Now denote the tangent vector to λ_k at P by $\left(\frac{\partial}{\partial x^k}\right)_P$, i.e.,

$$\left(\frac{\partial}{\partial x^k}\right)_P f = \dot{\lambda}_k(f)\Big|_P = \frac{d}{dt}\left(f \circ \lambda_k\right)\Big|_P . \tag{4.5}$$

This notation makes sense when we remember Eq. (3.9). Using it we can write

$$\dot{\lambda}_k(f)\Big|_P = \left(\frac{\partial f}{\partial x^k}\right)_P \forall f \in C^\infty(\mathcal{M}). \tag{4.6}$$

Note that $\left(\frac{\partial}{\partial x^k}\right)_P$ is notation. We should understand this as

$$\left(\frac{\partial}{\partial x^k}\right)_P f = \frac{\partial}{\partial x^k}\left(f \circ \varphi^{-1}\right)\Big|_{\varphi(P)} \equiv \frac{\partial f}{\partial x^k}\Big|_{\varphi(P)} \tag{4.7}$$

in a chart around P. The $\left(\frac{\partial}{\partial x^k}\right)_P$ are defined only when this chart is given, but these are vectors on the manifold at P, not on $\mathbb{R}^n$.

Let us now show that the tangent space at P has $\dot{\lambda}_k|_P$ as a basis. Take any vector $v_P \in T_P\mathcal{M}$, which is the tangent vector to some curve γ at P. (We may sometimes refer to P as $\gamma(0)$ or as $t = 0$.) Then

$$v_P(f) = \frac{d}{dt}\left(f \circ \gamma\right)\Big|_{t=0} \tag{4.8}$$

$$= \frac{d}{dt}((f \circ \varphi^{-1})\Big|_{\varphi(P)} \circ (\varphi \circ \gamma))\Big|_{t=0} . \tag{4.9}$$

Note that $\varphi \circ \gamma : I \to \mathbb{R}^n$, $t \mapsto (x^1(\gamma(t)), \cdots, x^n(\gamma(t)))$ are the coordinates of the curve γ, so we can use the chain rule of differentiation to write

$$v_P(f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\Big|_{\varphi(P)} \frac{d}{dt}(x^i \circ \gamma)\Big|_{t=0} \tag{4.10}$$

$$= \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\Big|_{\varphi(P)} v_P(x^i) . \tag{4.11}$$

The first factor is exactly as shown in Eq. (4.7), so we can write

$$v_P(f) \quad = \quad \left(\frac{\partial}{\partial x^k}\right)_P f v_P(x^i) \qquad \forall f \in C^\infty(\mathcal{M}) \tag{4.12}$$

i.e., we can write

$$v_P = v_P^i \left(\frac{\partial}{\partial x^k}\right)_P \qquad \forall v_P \in T_P\mathcal{M} \tag{4.13}$$

where $v_P^i = v_P(x^i)$. Thus the vectors $\left(\frac{\partial}{\partial x^k}\right)_P$ span $T_P\mathcal{M}$. These are to be thought of as tangents to the coordinate curves in φ. These can be shown to be linearly independent as well, so $\left(\frac{\partial}{\partial x^k}\right)_P$ form a basis of $T_P\mathcal{M}$ and v_P^i are the components of v_P in that basis.

The $\left(\frac{\partial}{\partial x^k}\right)_P$ are called **coordinate basis vectors** and the set $\left\{\left(\frac{\partial}{\partial x^k}\right)_P\right\}$ is called the **coordinate basis**.

It can be shown quite easily that for any smooth (actually C^1) function f a vector v_P defines a derivation $f \mapsto v_P(f)$, i.e., satisfies linearity and Leibniz rule,

$$v_P(f + \alpha g) \quad = \quad v_P(f) + \alpha v_P(g) \tag{4.14}$$

$$v_P(fg) \quad = \quad v_P(f)g(P) + f(P)v_P(g) \tag{4.15}$$

$$\forall f, g \in C^1(\mathcal{M}) \text{ and } \alpha \in \mathbb{R} \tag{4.16}$$