Chapter 5

Dual space

• The ***dual space*** $T_P^*\mathcal{M}$ of $T_P\mathcal{M}$ is the space of linear mappings $\omega : T_P\mathcal{M} \to \mathbb{R}$. □

We will write the action of ω on $v_P \in T_P\mathcal{M}$ as $\omega(v_P)$ or sometimes as $\langle \omega \mid v_P \rangle$.

Linearity of the mapping ω means

$$\omega(u_P + av_P) = \omega(u_P) + a\omega(v_P)\,, \qquad \forall u_P, v_P \in T_P\mathcal{M} \text{ and } a \in \mathbb{R}\,. \tag{5.1}$$

The dual space is a vector space under the operations of vector addition and scalar multiplication defined by

$$a_1\omega_1 + a_2\omega_2 : v_P \mapsto a_1\omega_1(v_P) + a_2\omega_2(v_P)\,. \tag{5.2}$$

• The elements of $T_P^*\mathcal{M}$ are called ***dual vectors***, ***covectors***, ***cotangent vectors*** etc. □

A dual space can be defined for any vector space V as the space of linear mappings $V \to \mathbb{R}$ (or $V \to \mathbb{C}$ if V is a complex vector space).

Example:

Vector	Dual vector
column vectors	row vector
kets $\vert\psi\rangle$	bras $\langle\phi\vert$
functions	linear functionals, etc.□

• Given a function on a manifold $f : \mathcal{M} \to \mathbb{R}$, every vector at P produces a number, $v_P(f) \in \mathbb{R} \quad \forall v_P \in T_P\mathcal{M}$. Thus f defines a covector $\mathrm{d}f$, given by $\mathrm{d}f(v_P) = v_P(f)$ called the **differential** or **gradient** of f. □

Since v_P is linear, so is $\mathrm{d}f$,

$$\begin{aligned} \mathrm{d}f(v_P + aw_P) &= (v_P + aw_P)(f) \\ &= v_P(f) + aw_P(f) \\ &\qquad \forall v_P, w_P \in T_P\mathcal{M}, a \in \mathbb{R}\,. \end{aligned} \tag{5.3}$$

Thus $\mathrm{d}f \in T_P^*\mathcal{M}$.

Proposition: $T_P^*\mathcal{M}$ is also n-dimensional.

Proof: Consider a chart φ with coordinate functions x^i. Each x^i is a smooth function $x^i : \mathcal{M} \to \mathbb{R}$. then the differentials $\mathrm{d}x^i$ satisfy

$$\mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right)_P = \left(\frac{\partial}{\partial x^j}\right)_P (x^i) = \frac{\partial}{\partial x^j}\left(x^i \circ \varphi^{-1}\right)\Big|_{\varphi(P)} = \delta^i_j\,. \tag{5.4}$$

The differentials $\mathrm{d}x^i$ are covectors, as we already know. So we have constructed n covectors in $T_P^*\mathcal{M}$. Next consider a linear combination of these covectors, $\omega = \omega_i \mathrm{d}x^i$. If this vanishes, it must vanish on every one of the basis vectors. In other words,

$$\begin{aligned} \omega = 0 \Rightarrow \omega\left(\frac{\partial}{\partial x^j}\right)_P &= 0 \\ \Rightarrow \omega_i \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right)_P &= 0 \\ \Rightarrow \omega_i \delta^i_j = 0 \quad &i.e. \quad \omega_j = 0\,. \end{aligned} \tag{5.5}$$

So the $\mathrm{d}x^i$ are linearly independent.

Finally, given any covector ω, consider the covector $\lambda = \omega - \omega\left(\frac{\partial}{\partial x^i}\right)_P \mathrm{d}x^i$. Then letting this act on a coordinate basis vector, we get

$$\begin{aligned} &\lambda\left(\frac{\partial}{\partial x^j}\right)_P \\ &\qquad = \omega\left(\frac{\partial}{\partial x^j}\right)_P - \omega\left(\frac{\partial}{\partial x^i}\right)_P \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)_P \\ &\qquad = \omega\left(\frac{\partial}{\partial x^j}\right)_P - \omega\left(\frac{\partial}{\partial x^i}\right)_P \delta^i_j = 0 \forall j \end{aligned} \tag{5.6}$$

So λ vanishes on all vectors, since the $\left(\frac{\partial}{\partial x^j}\right)_P$ form a basis. Thus the $\mathrm{d}x^i$ span $T_P^*\mathcal{M}$, so $T_P^*\mathcal{M}$ is n-dimensional.

Also, as we have just seen, any covector $\omega \in T_P^*\mathcal{M}$ can be written as

$$\omega = \omega_i \mathrm{d}x^i \qquad \text{where} \quad \omega_i = \omega\left(\frac{\partial}{\partial x^i}\right)_P , \tag{5.7}$$

so in particular for $\omega = \mathrm{d}f$, we get

$$\omega_i \equiv (\mathrm{d}f)_i = \mathrm{d}f\left(\frac{\partial}{\partial x^i}\right)_P = \left(\frac{\partial f}{\partial x^i}\right)_{\varphi(P)} \tag{5.8}$$

This justifies the name gradient.

It is straightforward to calculate the effect of switching to another overlapping chart, i.e. a coordinate transformation. In a new chart φ' where the coordinates are y^i (and the transition functions are thus $y^i(x)$) we can use Eq. (5.8) to write the gradient of y^i as

$$\mathrm{d}y^i = \left(\frac{\partial y^i}{\partial x^j}\right)_P \mathrm{d}x^j \tag{5.9}$$

This is the result of coordinate transformations on a basis of covectors.

Since $\left\{\left(\frac{\partial}{\partial x^i}\right)_P\right\}$ is the dual basis in $T_P\mathcal{M}$ to $\{\mathrm{d}x^i\}$, in order for $\left\{\left(\frac{\partial}{\partial y^i}\right)_P\right\}$ to be the dual basis to $\{\mathrm{d}y^i\}$ we must have

$$\left(\frac{\partial}{\partial y^i}\right)_P = \left(\frac{\partial x^j}{\partial y^i}\right)_P \left(\frac{\partial}{\partial x^j}\right)_P \tag{5.10}$$

These formulae can be generalized to arbitrary bases.

Given a vector v, it is not meaningful to talk about its dual, but given a basis $\{e_a\}$, we can define its dual basis $\{\omega^a\}$ by $\omega^a(e_b) = \delta^a_b$.

We can make a change of bases by a linear transformation,

$$\omega^a \mapsto \omega'^a = A^a_b \omega^b , \qquad\qquad e_a \mapsto e'_a = (A^{-1})^b_a e_b , \tag{5.11}$$

with A a non-singular matrix, so that $\omega'^a(e'_b) = \delta^a_b$.

Given a 1-form λ we can write it in both bases,

$$\lambda = \lambda_a \omega^a = \lambda'_a \omega'^a = \lambda'_a A^a_b \omega^a , \tag{5.12}$$

from which it follows that $\lambda'_a = (A^{-1})^b_a \lambda_b$.

Similarly, if v is a vector, we can write

$$v = v^a e_a = v'^a e'_a = v'^a (A^{-1})^b_a e_b \,, \tag{5.13}$$

and it follows that $v^a = A^a_b v^b$.

• Quantities which transform like λ_a are called **covariant**, while those transforming like v^a are called **contravariant**. □