

Novel Temporal Behavior of a Nonlinear Dynamical System: The Completely Inelastic Bouncing Ball

Anita Mehta⁽¹⁾ and J. M. Luck^(1,2)

⁽¹⁾*Theory of Condensed Matter, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, England*

⁽²⁾*Service de Physique Théorique, Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique, Centre d'Etudes Nucléaires de Saclay, 91191 Gif-sur-Yvette CEDEX, France*

(Received 14 February 1990)

We consider a ball under the influence of gravity on a vibrating platform where the ball-platform collisions are completely inelastic. We present for the first time several remarkable features in the temporal behavior of the system; its phase space is divided into transmitting and absorbing regions, which are responsible for the abrupt termination of a period-doubling sequence and the onset of a locking regime, in which an appropriately defined winding number has intriguing scaling properties.

PACS numbers: 05.45.+b, 02.50.+s

The dynamical system that we study here is as beautiful in its simplicity of conception as it is in the rich complexity of its behavior—it consists of a ball on a harmonically vibrating platform such as a loudspeaker, whose height at time t is given by

$$s(t) = A \sin \omega t. \quad (1)$$

The coefficient of restitution for the ball-platform collisions α determines the nature of the observed dynamics. The original elastic version of the problem with $\alpha = 1$ was introduced by Fermi,¹ and many approximations to the problem have since been studied experimentally and theoretically.²⁻⁹

However, our approach to this problem is a first in many senses. In our treatment of the completely inelastic case ($\alpha = 0$), we construct what seems to be the first consistent physical approximation to the exact map, demonstrate that the former is in itself a novel and non-trivial dynamical system with transmitting and absorbing regions of phase space, show by a scaling analysis for the exact and approximate cases that their temporal behaviors are qualitatively the same, and present our rather intriguing results for these. The latter consist of the discontinuous interruption of a period-doubling sequence and its replacement by a locking regime where a suitably defined winding number exhibits very singular scaling properties.

The exact mapping for the case of a completely inelastic ball can be easily derived from Newton's equation:

$$A(\sin \omega t - \sin \omega t_k) + \frac{1}{2} g(t - t_k)^2 - (A\omega \cos \omega t_k)(t - t_k) = 0, \quad (2)$$

where g is the gravitational acceleration. If the ball is launched by the platform at time t_k , the bouncing condition $A\omega^2 \sin \omega t_k > g$ being obeyed, it will land at the first time $t > t_k$ which fulfills Eq. (2). The ball will then be relaunched as soon as the above bouncing condition is satisfied, namely, either immediately or in the beginning of the next vibration cycle. We decide at this point, in

view of the complexity of the implicit equation (2) and in keeping with tradition,² to construct an approximate map which reflects the behavior of the real system. However, *out of* keeping with tradition, our map avoids unphysical consequences like jumps backward in time.¹⁰

The physical basis of our approximate map is that, for $A\omega^2$ sufficiently larger than g , the first (transmitting) quadrant of the vibration cycle is the only one where the ball bounces off the platform, and has an appreciable trajectory in air before its next collision. An analysis of the exact map shows that if the ball is in one of the other (absorbing) three, it does one of two things: It either waits on the platform until the next transmitting interval, or returns to the platform before the next transmitting interval so that in both cases it is relaunched in the next transmitting interval. This is a consequence of the coefficient of restitution of the ball being precisely zero, so that the ball loses all memory of its earlier velocity when it hits the platform. In this spirit, we present our approximate map:

$$\tau_{k+1} = \begin{cases} \tau_k + \Gamma \cos 2\pi \tau_k, & \text{for } \text{Frac}(\tau_k) < \frac{1}{4}, \\ \text{Int}(\tau_k + 1), & \text{otherwise,} \end{cases} \quad (3a)$$

$$\tau_{k+1} = \begin{cases} \tau_k + \Gamma \cos 2\pi \tau_k, & \text{for } \text{Frac}(\tau_k) < \frac{1}{4}, \\ \text{Int}(\tau_k + 1), & \text{otherwise,} \end{cases} \quad (3b)$$

with $\tau_k = \omega t_k / 2\pi$ and $\Gamma = A\omega^2 / \pi g$. Here and throughout the following, $\text{Int}(x)$ and $\text{Frac}(x)$ denote the integer and fractional parts of a real quantity x , respectively. The dimensionless reduced acceleration Γ will be the key parameter of our analysis. Thus, Eq. (3b) models the physical situation in the absorbing region, while Eq. (3a) can be derived as a large- Γ approximation to Eq. (2) by assuming that $\tau_{k+1} - \tau_k \sim \Gamma$ and keeping terms of highest order in Γ .

The n -periodic solution $\tau_{k+1} = \tau_k + n$ has a region of stability that can be shown to be $n \leq \Gamma \leq (n^2 + 1/\pi^2)^{1/2}$; we now span the temporal behavior of the ball as a function of Γ , starting from any integer value $\Gamma = n$. This can be done by writing down a scaling form valid for Γ close to a large integer n , which we present for the approxi-

mate [Eqs. (3)] and exact [Eq. (2)] maps, respectively:

$$X' = X + \lambda - 2\pi^2 X^2, \quad X > 0, \quad (4a)$$

$$X' = X + \lambda/2 - \pi^2 X^2, \quad X > 1/2\pi^2, \quad (4b)$$

where $X/n = \text{Frac}(\tau)$ and $\lambda/n = \text{Frac}(\Gamma)$. The scaling form of both maps is thus a quadratic transform, where λ measures the amount of nonlinearity. For the approximate map [Eq. (4a)], the condition $X > 0$ ceases to be fulfilled by the attractor for $\lambda \approx 0.08208$, soon after the eight-cycle sets in. A similar discontinuous instability takes place with the two-cycle for Eq. (4b). What this means physically is that the period-doubling cascade is interrupted by the ball's landing in an absorbing region. It is worth remarking that the termination of the period-doubling sequence is entirely due to the fact that the map is defined in a *piecewise* fashion, so that it assumes two different forms, according to inequalities given explicitly in Eqs. (3a) and (3b). (This piecewise structure also belongs, in a qualitative way, to the exact map.) Thus while the *effects* are in some sense halfway between the behavior of one- and two-dimensional maps, the actual map itself acts (in a discontinuous way) on *one* variable, namely, X .

The interruption of the period-doubling sequence suggests that from this point onward in Γ space, the ball will generically, after sufficiently many iterations, land in an absorbing region characterized by Eq. (3b)—its motion is then periodic. We call this the locking region, and devise a separate framework to explore its properties.

Assume the initial condition is $t_0 = 0$. Let l be the smallest integer such that $\text{Frac}(t_l) > \frac{1}{4}$, and m be the integer part of t_l : We then define the winding number Ω as the mean duration of a ball's flight in air, in units of the period $2\pi/\omega$ of the platform, so that

$$\Omega = \Omega_{m,l} = (m+1)/(l+1). \quad (5)$$

Thus Ω is rational in every locked interval. At fixed n , we have the bounds $1 \leq \Omega_{m,l} \leq (n+1)/2$. If we now write $\Gamma = n + f$, with $\text{Int}(\Gamma) = n$ and $\text{Frac}(\Gamma) = f$, then starting with $\tau_0 = 0$ (so that $\tau_1 = \Gamma$), we see immediately that

$$l = 1, \quad m = n, \quad \Omega = (n+1)/2, \quad \text{for } f > \frac{1}{4}, \quad (6)$$

$$\tau_2 = n + f + (n+f)\cos 2\pi f, \quad \text{for } f < \frac{1}{4}.$$

For $k \geq 2$, an analogous equation holds with $\tau_k = m + \tau_k^f$:

$$\tau_{k+1} = (m + \tau_k^f) + (n+f)\cos 2\pi \tau_k^f, \quad (7)$$

where m and τ_k^f are, respectively, the integer and fractional parts of τ_k . This is sketched in Fig. 1: A series of "windows" on the τ_k^f axis give rise to the absorption bands in τ_{k+1} which are the crucial ingredients in the scaling that we will describe. It can be seen fairly readily that between any two consecutive bands (m,l) and

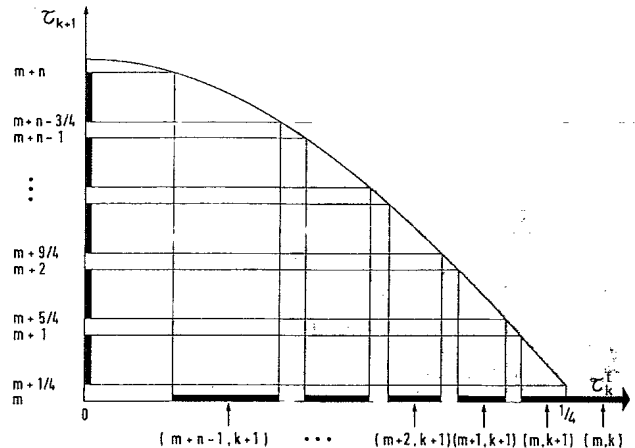


FIG. 1. Sketch of τ_{k+1} vs τ_k^f ($k \geq 2$) for $n = \text{Int}(\Gamma) = 5$, to show the windows in τ_k^f that lead to absorption bands in τ_{k+1} : These bands are indexed by winding numbers $\Omega_{m,l}$.

$(m-1,l)$, there will be a series of n new bands at the next step, labeled from $(m,l+1)$ to $(m+n-1,l+1)$, where (m,l) and $(m,l+1)$ are contiguous, while $(m-1,l)$ and $(m+n-1,l+1)$ are not.

Figure 2 illustrates the rich behavior of the winding number Ω —a quantity accessible to experiment in principle—induced by the infinite band nesting just described. (Despite appearances due to graphical constraints, there is a *single* Γ for each Ω —so that the small regions in the devil's-staircase-like structure do *not* overlap.) We speculate here that the winding number Ω might assume irrational (unlocked) values for some Cantor set of values of Γ , and that Ω is not a continuous function of Γ in general. This is because the iterative construction of intervals of Γ where the motion is locked

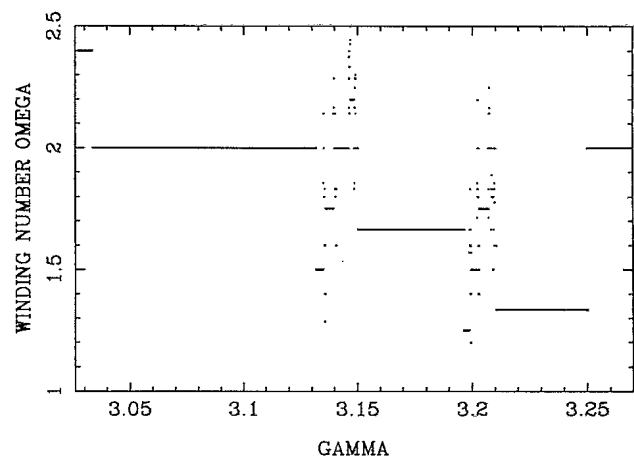


FIG. 2. Plot of the rotation number Ω vs Γ for $n = \text{Int}(\Gamma) = 3$: Each absorption interval corresponds to a rational winding number $\Omega = \Omega_{m,l}$. Despite appearances due to graphical constraints, there is a *single* Γ for each Ω , so that the small regions in the devil's-staircase-like structure do *not* overlap.

leaves increasing numbers of progressively smaller gaps under iteration, where Ω remains unpredicted. It seems plausible that these gaps build up a nonempty limit set, with possibly a Cantor-like structure. Since Ω is not a monotonic function of Γ when it is restricted to the locked intervals (see Fig. 2), it is unlikely that it will become a continuous function $\Omega(\Gamma)$ when the gaps on the limit set are added.

In order to probe, more quantitatively, the nature of the scaling of these absorption regions under iteration, we define for each generation k the partition function

$$Z_k(q) = \sum_{i=1}^{N_k} (\lambda_i^{(k)})^q, \quad N_k = n^k, \quad (8)$$

$$\frac{Z_{k+1}}{Z_k} \cong S(q) = \left[\frac{2}{3\pi} \right]^{q-1} \sum_{p=0}^{n-1} \left[\cos^{-1} \left(\frac{p + \frac{1}{4}}{n+f} \right) - \cos^{-1} \left(\frac{p+1}{n+f} \right) \right]^q \quad (11)$$

and hence $Z_k(q) \sim [S(q)]^k$. This scaling relation is crucial in determining the scaling properties of the distribution of the λ 's. It can be shown that in the large- Γ limit, its successive moments are given by

$$\mu_1 = \langle \lambda^{(k)} \rangle \sim (1/4n)^k, \quad (12a)$$

$$\mu_2 = \langle [\lambda^{(k)}]^2 \rangle \sim (\ln n / 8\pi^2 n^2)^k, \quad (12b)$$

$$\mu_q = \langle [\lambda^{(k)}]^q \rangle \sim [B(f, q) n^{-1-q/2}]^k, \quad q \geq 3, \quad (12c)$$

where $B(f, q)$ is nontrivial as a function of both f and q . We have thus $\mu_q \gg \mu_1^q$ for $q \geq 2$. This behavior signals an appreciable population of anomalously long locking intervals, generated by the square-root singularity of the arccosine function entering the dispersion curves (10). The results (12) can be extended to real values of the moment index q : One has $Z_k(q) \sim N_k^{-(q-1)D_q}$, with generalized dimensions $D_q = 1$ for $q < 2$ and $D_q = q/2(q-1)$ for $q > 2$. The problem thus exhibits a partial multifractal spectrum of generalized dimensions.

We have also checked¹¹ that all the features mentioned above hold in a semiquantitative sense for the exact map (2): We expect there too the occurrence of such a locking region, from the global structure of the map. Although we postpone to a later paper¹¹ a detailed quantitative comparison of the exact map and our approximate one, we have asserted here the equivalence of their behavior in the scaling region, via Eqs. (4a) and (4b). In each case, there is the onset of an instability which interrupts the period-doubling cycles: This occurs in the approximate map during the eight-cycle when $X < 0$, and for the exact map during the two-cycle when $X < 1/2\pi^2$, the corresponding values of λ in each case are trivially obtained from Eqs. (4a) and (4b). Thus in *both* maps, the period-doubling cascade is interrupted by the same mechanism, viz., the ball's landing in an absorbing region.

which involves the number of absorption regions N_k and their lengths $\lambda_i^{(k)}$ (in Γ space) at the k th stage of iteration. Each $\lambda_{i,p}^{(k+1)}$ at the next stage is generated by a window of fractional length δ_p in the *transmitting* region adjacent to $\lambda_i^{(k)}$, of length $\approx \frac{1}{3}\lambda_i^{(k)}$. We have thus

$$\lambda_{i,p}^{(k+1)} \approx \frac{4}{3}\lambda_i^{(k)}\delta_p, \quad 0 \leq p \leq n-1. \quad (9)$$

The δ_p are given to a very good approximation by

$$\delta_p = (1/2\pi) \{ \cos^{-1}[(p + \frac{1}{4})/(n+f)] - \cos^{-1}[(p+1)/(n+f)] \}. \quad (10)$$

These equations lead to the recursion formula

With regard to future work, we have shown already¹¹ that for α infinitesimally greater than zero, Eq. (4) becomes the Hénon map, which exhibits chaos—this is in accord with experimental observations⁷ for the bouncing ball with partial elasticity. We hope now to analyze the $\alpha > 0$ case by the methods described in this Letter—this work, along with details of the solution of the exact map (2), will form the subject of a later paper.¹¹

In conclusion, we have shown that the apparently simple system of a completely inelastic ball bouncing on a vibrating platform exhibits a remarkable complexity in its temporal behavior: We have devised an excellent approximate map (which replicates qualitatively the behavior of the exact map¹¹), and shown that its period-doubling sequence terminates abruptly in a locking region. Finally, we have demonstrated that in this regime, the intervals in Γ space characterized by different rational winding numbers Ω show intriguing scaling behavior, with moments growing according to an explicit partial multifractal law.

Most of this work was done during a visit by J.M.L. to the Cavendish Laboratory, Cambridge, where he was supported by the French-British prize. A.M. acknowledges the support of the Science and Engineering Research Council, United Kingdom, and the hospitality of the Service de Physique Théorique, Saclay; she is grateful to Professor P. G. de Gennes for introducing her to this problem, and to Pierre Evesque for references to the existing literature. We thank Richard Needs for a careful reading of the manuscript.

¹E. Fermi, Phys. Rev. **75**, 1169 (1949).

²L. D. Pustyl'nikov, Trans. Moscow Math. Soc. **2**, 1 (1978).

³G. M. Zaslavskii, Phys. Lett. **69A**, 145 (1978).

- ⁴B. V. Chirikov, *Phys. Rev.* **52**, 263 (1979).
- ⁵A. J. Lichtenberg, M. A. Lieberman, and R. H. Cohen, *Physica (Amsterdam)* **1D**, 291 (1980).
- ⁶J. Guckenheimer and P. J. Holmes, *J. Sound Vib.* **84**, 173 (1982).
- ⁷P. Pieranski, *J. Phys. (Paris)* **44**, 573 (1983); P. Pieranski, Z. J. Kowalik, and M. Franaszek, *J. Phys. (Paris)* **46**, 681 (1985).
- ⁸R. M. Everson, *Physica (Amsterdam)* **19D**, 355 (1986).
- ⁹K. Wiesenfeld and N. B. Tuffillaro, *Physica (Amsterdam)* **26D**, 321 (1987).
- ¹⁰See, e.g., Refs. 6 and 7; Ref. 8 comments on the issue of negative times as well, but gets around this problem in a somewhat contrived way.
- ¹¹Anita Mehta and J. M. Luck, Centre d'Etudes Nucléaires de Saclay report, 1990 (to be published).