How the rich get richer

Anita Mehta[1], A S Majumdar[1] and J M Luck[2]

[1] S. N. Bose National Centre for Basic Sciences, Block JD, Salt Lake, Kolkata 700 098, India
[2] Service de Physique Theorique, CEA Saclay, F 91191 Gif-sur-Yvette, France
anita@bose.res.in, archan@bose.res.in, luck@spht.saclay.cea.fr

Summary. In our model, n traders interact with each other and with a central bank; they are taxed on the money they make, some of which is dissipated away by corruption. A generic feature of our model is that the richest trader always wins by 'consuming' all the others: another is the existence of a *threshold* wealth, below which all traders go bankrupt. The two-trader case is examined in detail, in the *socialist* and *capitalist* limits, which generalise easily to $n > 2$. In its mean-field incarnation, our model exhibits a two-time-scale *glassy dynamics*, as well as an astonishing *universality*. When preference is given to local interactions in finite neighbourhoods, a novel feature emerges: instead of at most one overall winner in the system, finite numbers of winners emerge, each one the *overlord* of a particular region. The patterns formed by such winners (*metastable states*) are very much a consequence of initial conditions, so that the fate of the marketplace is ruled by its past history; *hysteresis* is thus also manifested.

1 Introduction

The tools of statistical mechanics [1] are increasingly being used to analyse problems of economic relevance [2]. Our model below, although originally formulated to model the evolution of primordial black holes [3, 4], is an interesting illustration of the *rich-get-richer* principle in economics. It is inherently disequilibrating; individual traders interact in such a way that the richest trader *always* wins.

2 The model

In this model, n traders are linked to each other, as well as to a federal reserve bank; an individual's money accrues interest at the rate of $\alpha > 1/2$ [3] but is also taxed such that it is depleted at the rate of $1/t$, where t is the time. The interaction strength g_{ij} between traders i and j is a measure of how much of their wealth is invested in trading; income from trading is also taxed at the rate of $t^{1/2}$. There is a *threshold* term such that the less a trader has, the more he loses; additionally the model is *non-conservative* such that some of the wealth disappears forever from the local economy. These last terms can have different interpretations in a macro- or a micro-economic context. In the former case (where the traders could all be citizens of a country linked by a federal bank), the threshold term could represent the plight of the (vanishing) middle classes, while the non-conservative nature of the model could represent the contribution of *corruption* to the economy - some of the taxed money disappears forever from the region, to go either to the black economy or to foreign shores. In a more micro-economic context (where traders linked by a bank are a subset of the major economy), the interpretation is the reverse: the non-conservative nature of the model would imply money lost irretrievably by taxation (to go to social benefits from which the traders do not themselves benefit), while the threshold term could represent the effect of corruption (poorer traders lose more by graft than richer ones). Including all these features, we postulate that the wealth $m_i(t)$ for $i = 1, \ldots, n$ of each trader evolves as follows [4]:

$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = \left(\frac{\alpha}{t} - \frac{1}{t^{1/2}} \sum_j g_{ij} \frac{\mathrm{d}m_j}{\mathrm{d}t} \right) m_i - \frac{1}{m_i}. \tag{1}$$

In the following,we use units of reduced time $s = \ln \frac{t}{t_0}$ (to renormalise away the effect of initial time t_0), reduced wealth $x_i = \frac{m_i}{t^{1/2}}$ and reduced square wealth $y_i = x_i^2 = \frac{m_i^2}{t}$. In these units, we recall the result for an *isolated* trader [3]. A trader whose initial wealth y_0 is greater than $y_\star$, (with $y_\star(t_0) = \left(\frac{2t_0}{2\alpha-1}\right)$), is a *survivor* who keeps getting richer forever: a trader with below this threshold wealth goes bankrupt and disappears from the marketplace in a finite time. The influence of this initial threshold $y_\star$ will be seen to persist throughout this model: in every case we examine, surviving winners will all be wealthier than this.

3 A tale of two traders: socialist vs capitalist?

We examine the two-trader case in the *socialist* and *capitalist* limits. In the socialist limit, the initial equality of wealth is maintained forever by symmetry: their common wealth $x(s)$ obeys:

$$x' = \frac{(2\alpha - 1)x^2 - 2 - gx^3}{2x(1 + gx)}. \tag{2}$$

This equation is analytically tractable: it has fixed points given by $(2\alpha-1)x^2 - 2 - gx^3 = 0$. A critical value of the interaction strength g, $g_c = \left(\frac{2(2\alpha-1)^3}{27}\right)^{1/2}$, separates two qualitatively different behaviours. For $g > g_c$, there is no fixed point; overly heavy trading (insufficient saving) causes both traders to go quickly bankrupt, independent of their initial capital. In the opposite case of sensible trading, $g < g_c$, there are two positive fixed points, $y_\star^{1/2} < x_{(1)}$ (unstable) $< (3y_\star)^{1/2} < x_{(2)}$ (stable). If both traders are initial equally poor with wealth $x_0 < x_{(1)}$, this is dynamically attracted by $x = 0$ – the traders go rapidly bankrupt! For initially rich traders with $x_0 > x_{(1)}$, their wealth is dynamically attracted by $x_{(2)}$ – they grow richer forever as $m(t) \approx x_{(2)} t^{1/2}$, a growth rate which is *less* than that for an isolated trader! This case, where equality and overall prosperity prevail even though there are no individual winners, could correspond to a (modern) Marxist vision.

In the *capitalist* case, with traders who are initially unequally wealthy, any small differences always diverge exponentially early on: the details of this transient behaviour can be found in [5]. However, the asymptotic behaviour is such that richer trader wins, while the poorer one goes bankrupt: *the survival of the richest is the single generic scenario for two unequally wealthy traders.* At this point, we are back to the case of an isolated trader referred to in Section 3: he may, depending on whether his wealth at this point is less or greater than $y_\star$, also go bankrupt or continue to get richer forever.

All of the above generalises easily to any finite number $n \geq 2$ of interacting traders.

4 Infinitely many traders in a soup - the mean field limit

We now examine the limit $n \to \infty$: we first explore the *mean field behaviour* where every trader is connected to every other by the same dilute interaction $g = \frac{\overline{g}}{n}$. For fixed $\overline{g}$, the limit $n \to \infty$ leads to the *mean field equations* [5]:

$$y'(s) = \gamma(s) y(s) - 2 \tag{3}$$

When additionally, $\overline{g}$ is small (weak trading), a *glassy* dynamics [1] with two-step relaxation is observed. In Stage I, individual traders behave as if they were isolated, so that the survivors are richer than threshold ($y_\star$), exactly as in the one-trader case of Section 2. In Stage II, all traders interact *collectively*, and *slowly* [5]. All but the richest trader eventually go bankrupt during this stage.

The model also manifests a striking *universality*. For example, with an exponential distribution of initial wealth, the survival probability decays asymptotically as $S(t) \approx \frac{2\alpha-1}{\overline{g}} \left(C \ln \frac{t}{t_0} \right)^{-1/2}$; additionally, the mean wealth of the surviving traders grows as $\langle\!\langle m \rangle\!\rangle_t \approx \left(C\, t \ln \frac{t}{t_0} \right)^{1/2}$. In both cases, $C = \pi$ *irrespective* of α, $\overline{g}$ and the parameters of the exponential distribution. The universality we observe goes further than this, in fact: it can be shown [5] that C only depends on whether the initial distribution of wealth is bounded or not and on (the shape of) the tail of the wealth distribution.

5 Infinitely many traders with *local interactions* - the emergence of overlords

Still in the $n \to \infty$ limit, we now introduce local interactions: traders interact preferentially with their $z = 2D$ nearest neighbours on a D-dimensional lattice: once again we look at the limit of weak trading ($g \ll 1$). The dynamics once again consists of two successive well-separated stages with fast individual Stage I dynamics, whose survivors are richer than threshold, exactly as before (Section 4). The effects of going beyond mean field are only palpable in Stage II: the effect of local interactions lead to a slow dynamics which is now very different from the mean-field scenario above. The survival probability $S(s)$ in fact decays from its plateau value $S_{(1)}$ (number of Stage I survivors) to a non-trivial limiting value $S_{(\infty)}$; unlike the mean field result, a *finite* fraction of traders now survive forever!

Figure 1 illustrates this two-step relaxation in the decay of the survival probability $S(s)$. While the (non-interacting) decay to the plateau at $S_{(1)} = 0.8$ is (rightly) independent of g, the Stage II relaxation shows *ageing*; the weaker the interaction, the longer the system takes to reach the (non-trivial) limit survival probability $S_{(\infty)} \approx 0.4134$.

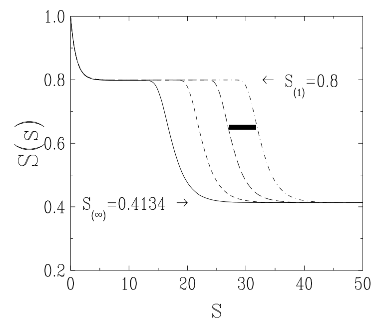

Fig. 1. Plot of the survival probability $S(s)$ on the chain with $S_{(1)} = 0.8$ (after reference [5]. Left to right: Full line: $g = 10^{-3}$. Dashed line: $g = 10^{-4}$. Long-dashed line: $g = 10^{-5}$. Dash-dotted line: $g = 10^{-6}$.

At the end of Stage II, the system is left in a non-trivial *attractor*, which consists of a pattern where each surviving trader is *isolated*, an *overlord* who keeps getting richer forever. We call these attractors *metastable states*, since they form valleys in the existing random energy landscape; the particular metastable state chosen by the system (corresponding to a particular choice of pattern) is the one which can most easily be reached in this landscape[1]. The number $\mathcal{N}$ of these states generically grows exponentially with the system size (number of sites) N as $\mathcal{N} \sim \exp(N\Sigma)$ with Σ the configurational entropy or *complexity*. The limit survival probability $S_{(\infty)}$ (Figure 1) is just the density of a typical attractor, i.e., the fraction of the initial clusters which survive forever.

We now examine in some more detail the fate of a set of $k \geq 1$ surviving traders: this depends on k as follows.

- ⋆ $k = 1$: If there is only one trader, he survives forever, trading with the reserve and getting richer.
- ⋆ $k = 2$: If a pair of neighbouring traders (represented as ●●) survive Stage I, the poorer dies out, while the richer is an overlord, leading to ●○ or ○●.
- ⋆ $k \geq 3$: If three or more traders survive Stage I, they may have more than one fate. Consider for instance (●●●): if the middle trader goes bankrupt first (●○●), the two end ones are isolated, and both will become overlords. If on the other hand the trader at the 'end' first goes bankrupt (e.g. ●●○), only the richer among them will become an overlord (e.g. ●○○). The pattern of these immortal overlords, and even their number, therefore *cannot* be predicted a priori.

Finally, we present some of the observed patterns. If $S_{(\infty)} = 1/2$ on, say, a square lattice, (i.e. the highest density of surviving traders is reached), there are only two possible 'ground-state' configurations of the system; the two possible patterns of immortal overlords are each perfect checkerboards of one of two possible parities. This allows for an interesting possibility: we can define a checkerboard index for each site, which classifies it according to its *parity* [5].

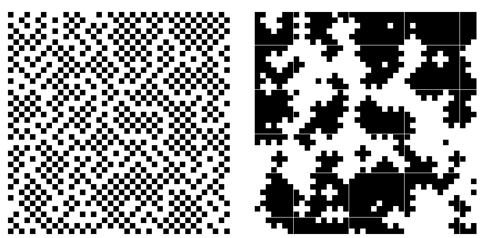

Fig. 2. Two complementary representations of a typical pattern of surviving clusters on a 40^2 sample of the square lattice, with $S_{(1)} = 0.9$, so that $S_{(\infty)} \approx 0.371$ (after reference [5]. **Left**: Map of the survival index. Black squares represent overlords for which $\sigma_{\mathbf{n}} = 1$, while white squares represent bankrupt sites for which $\sigma_{\mathbf{n}} = 0$. **Right**: Map of the checkerboard index. Black squares represent positive, while white squares represent negative, parity

Figure 2 shows a map of the survival index and of the checkerboard index for the same attractor for a particular sample of the square lattice. The *local* checkerboard structure, with random frozen-in defects between patterns of different parities is of course entirely inherited from the initial conditions. The overlords in the left-hand part of the figure are surrounded by rivulets of poverty ; in the right-hand figure, the deviation from a perfect checkerboard structure (all black or all white) is made clearer. Neighbouring sites are fully anticorrelated, because each overlord is surrounded by paupers: however, at least close to the limit $S_{(\infty)} = 1/2$, overlords are very likely to have next-nearest neighbours who are likewise overlords. The detailed examination of survival and mass correlation functions made in a longer paper [5] confirms these expectations.

To conclude, we have presented a model where traders interact through a reserve; we are able to model the effects of corruption and taxation via the non-conservative, threshold nature of our model. These could have different implications for micro- and macroeconomic situations. Our main results are that, in the presence of global interactions, typically only the wealthiest trader survives (provided he was born sufficiently rich); however, if traders interact locally, finite numbers of local overlords emerge by creating zones of poverty around them.

References

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