Dynamics of passive fields in turbulence

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Outline

- Introduction
- Dynamic multiscaling
- Summary of results
- Dynamics of Passive scalar
- Passive-vector

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- Introduction
- Dynamic multiscaling
  Dynamic Multiscaling in fluid turbulence
  - Multifractal Picture
  - Bridge relations
  - Eulerian vs Lagrangian
  - Ongoing Work
- Summary of results
- Dynamics of Passive scalar
- Passive-vector

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- Summary of results
- Dynamics of Passive scalar
  - Kraichnan model.
  - Simple dynamic scaling
  - Realistic advecting field
  - Dynamic multiscaling
- Passive-vector

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- Passive-vector
  - "Anti-Dynamo"
  - Dynamic multiscaling
  - Passive-vector shell model

Critical Phenomenon

Near the critical point
The spin-spin correlation function ($\Gamma(r, T_r)$):

$$\langle \phi(x)\phi(x+r) - \phi(x)\phi(x) \rangle \approx \frac{1}{r^{d-2+\eta}} \mathcal{F}(T_r^{\nu}\xi); \quad (0)$$

- $T_r \equiv (T - T_c)/T_c$
- zero external magentic field
- ξ correlation length, diverges at the critical point;
- η and ν : static critical exponents;
- $\mathcal{F}$: universal scaling function.

Scaling and Universality

- log-log plot of $\Gamma(r, T_r)$ vs r is a is a straight line slope $-d + 2 - \eta$.
- lattice-spacing$\ll r \ll$ system-size.
- η ν does not depend on details of the Hamiltonian.

Experiment

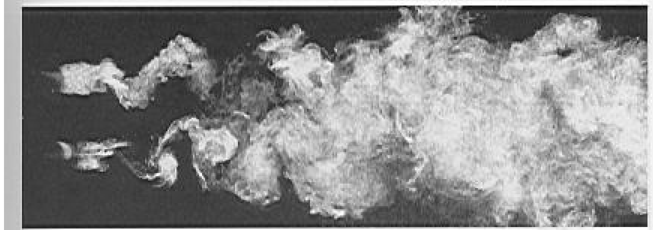


Fig. 1.10. Wake behind two identical cylinders at $R = 1800$. Courtesy R. Dumas.

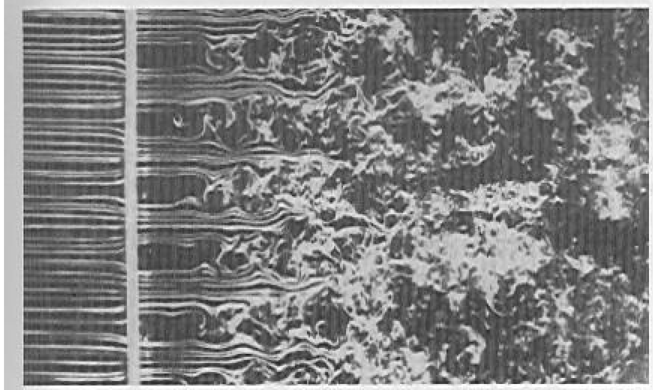
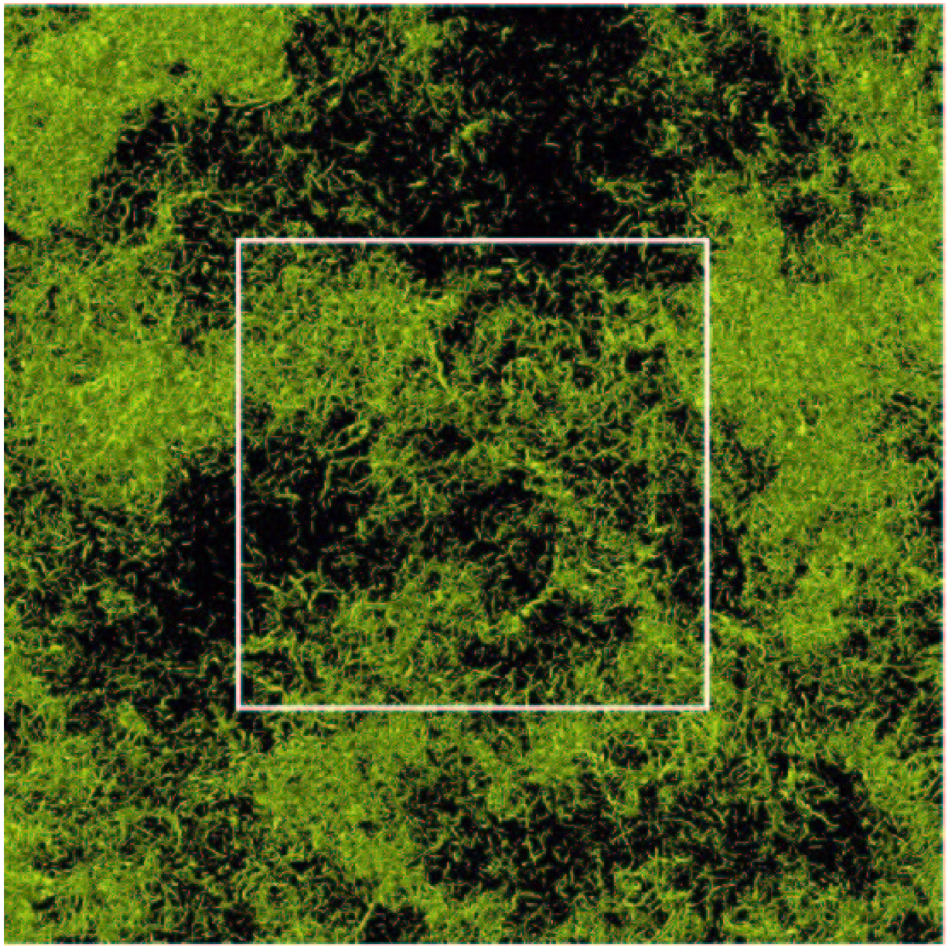
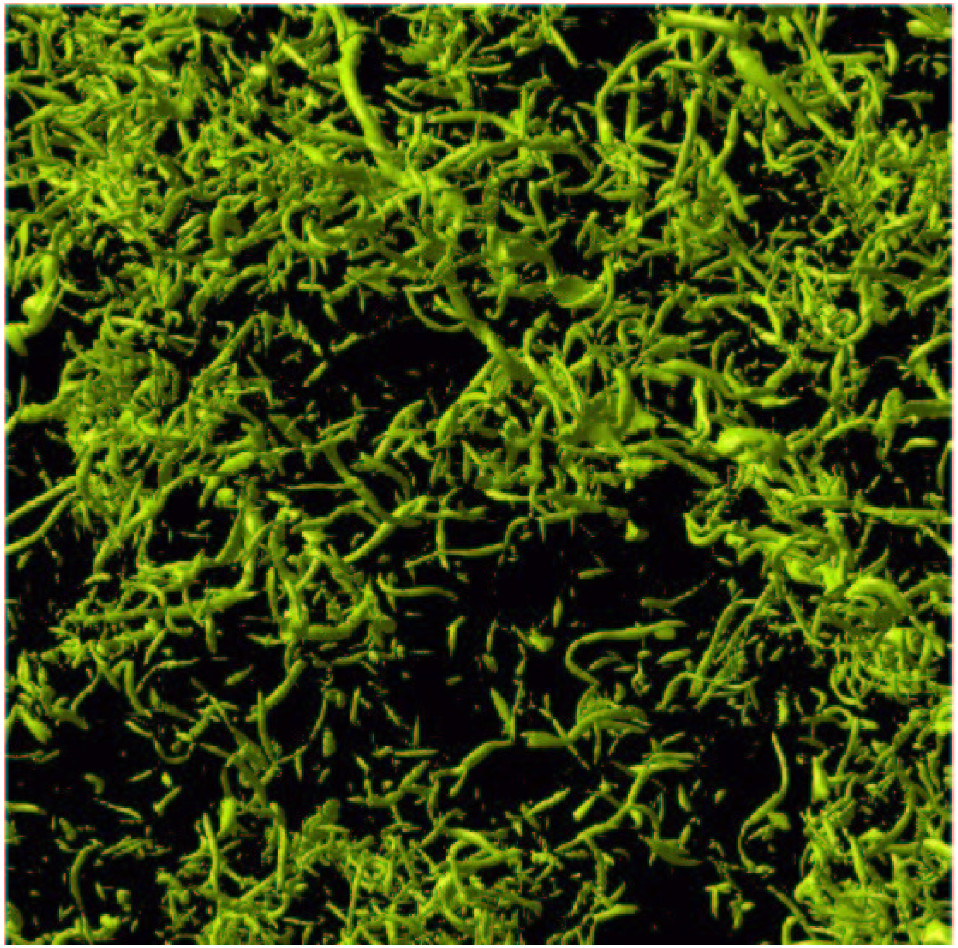


Fig. 1.11. Homogeneous turbulence behind a grid. Photograph T. Corke and H. Nagib.

Direct Numerical Simulation



Zoom-in



Introduction

Homogeneous and Isotropic Turbulence:

- Navier-Stokes equation:

$$\partial_t \vec{v} + (\vec{v}.\vec{\nabla})\vec{v} = \nu \nabla^2 \vec{v} + \vec{\nabla} p/\rho + \vec{f}/\rho,$$

$\vec{v}$, p, ν, and ρ are, respectively: Eulerian velocity, pressure, kinematic viscosity, and density; and $\vec{f}$ is the external forcing; incompressibility is imposed via $\vec{\nabla}.\vec{v} = 0$.

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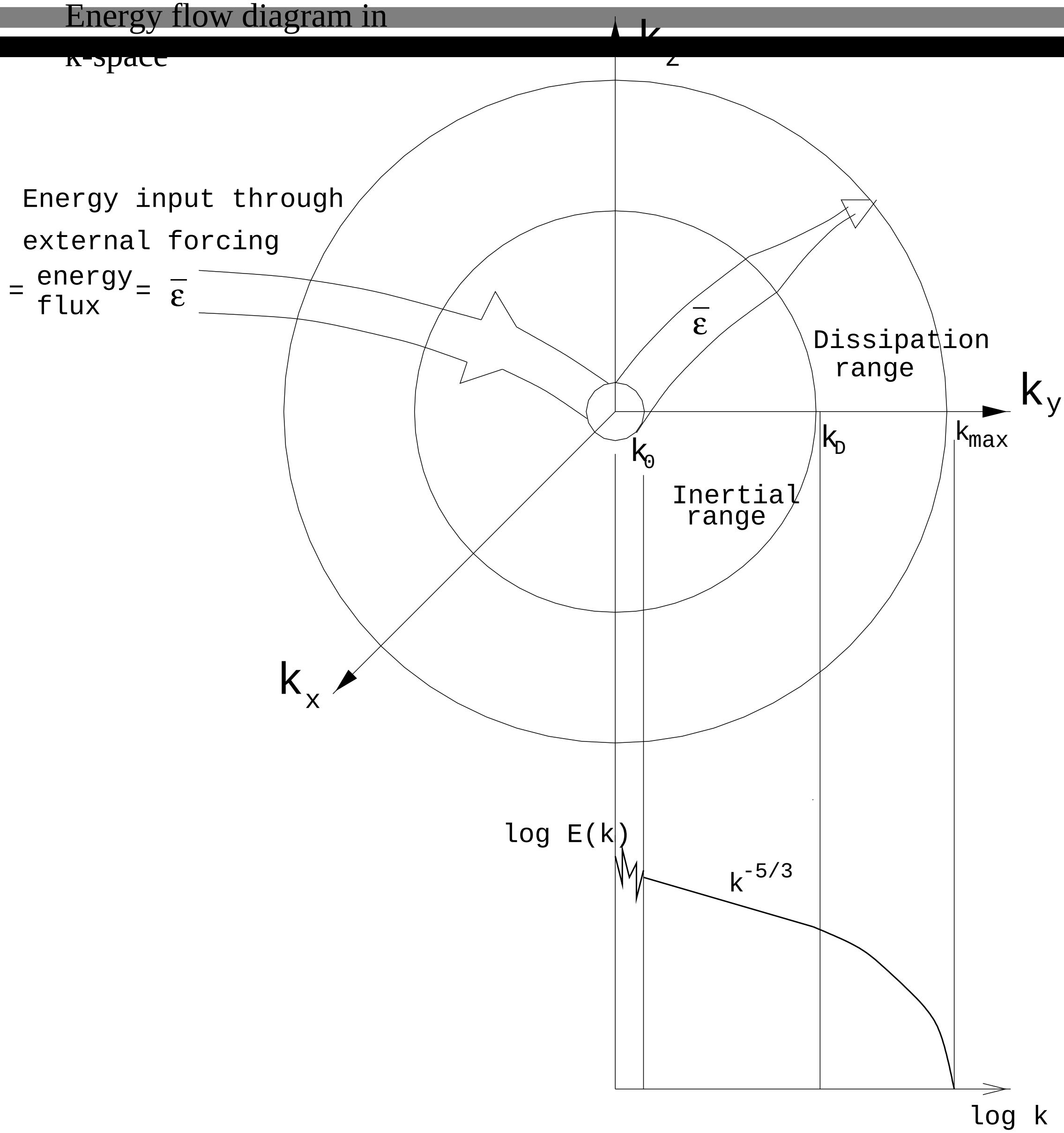
- In the *inertial range* , $\eta_d \ll \ell \ll L$, scaling exponents are independent of both the forcing term and the viscosity.(Universality)
- Energy cascades down the inertial range till it is dissipated in the dissipation range.

Cascade

Energy flow diagram in k-space



Energy spectrum in the steady state.

Structure functions

Order-p static, longitudinal velocity structure function.

$$
\begin{aligned}
S_p(\ell) &\equiv \langle \delta v(\ell, t)^p \rangle, \\
\delta v(\ell, t) &\equiv [\vec{v}(\vec{r} + \vec{\ell}, t) - \vec{v}(\vec{r}, t)] \cdot (\vec{\ell}/\ell),
\end{aligned}
$$

Multiscaling

- For ℓ in the *inertial range* we have

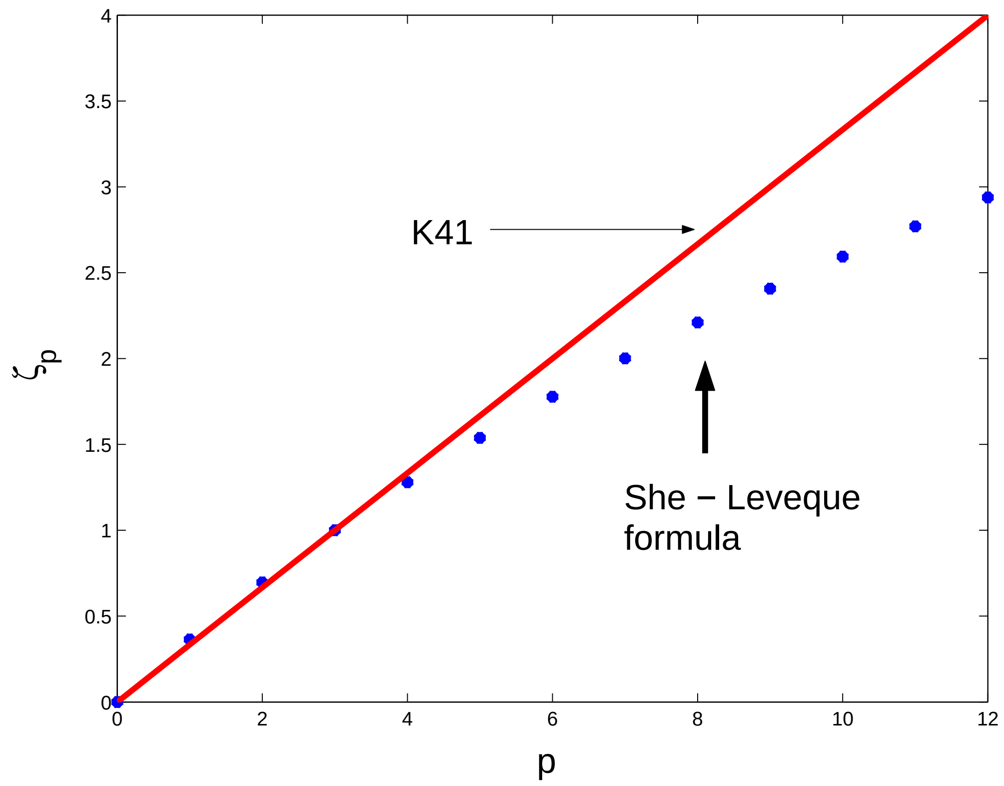
$$S_p(\ell) \sim \ell^{\zeta_p}.$$

- ζ_p is a nonlinear, convex function of p (Multiscaling) as in the She-Leveque formula

$$\zeta_p^{SL} = (p/9) + 2[1 - (2/3)^{p/3}],$$

which provides a reasonably good parametrization of experimental and numerical data for the multiscaling exponents ζ_p.

She-Leveque formula



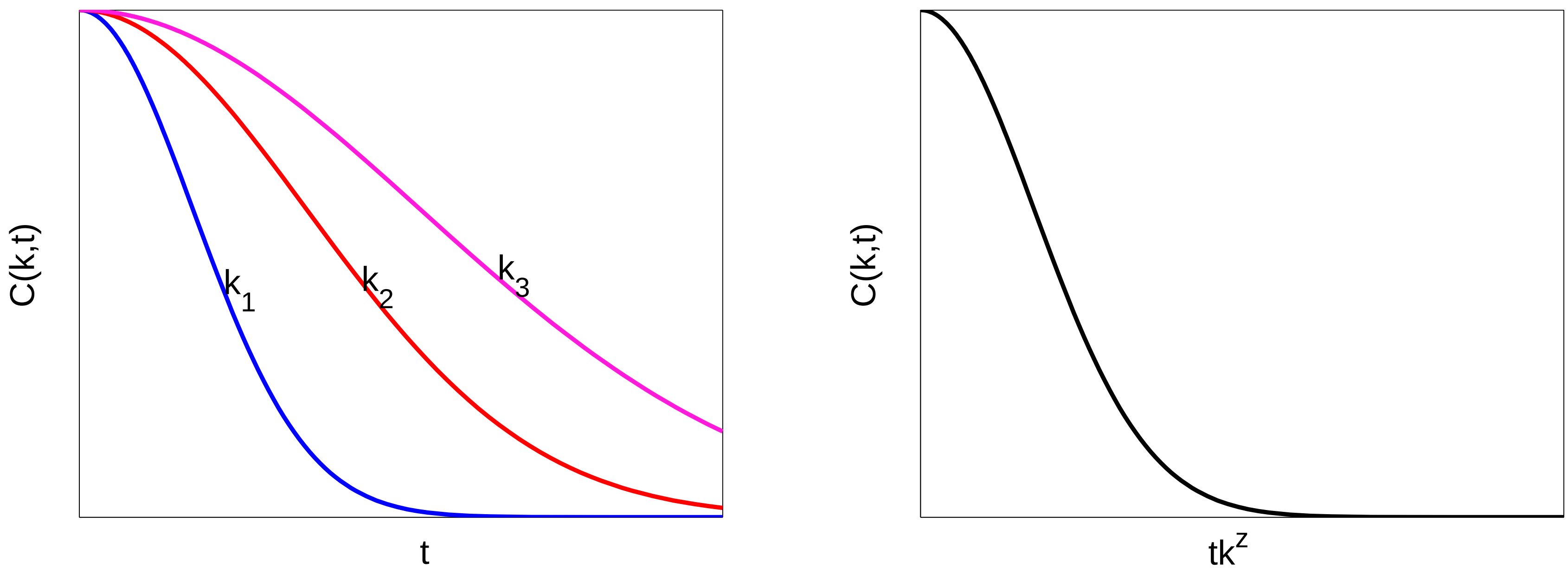
Exact relations

- $\zeta_3 = 1$ (von-Karmann-Howarth)
- ζ_p versus p is a convex curve.

Critical dynamics

- Near a critical point the relaxation time diverges.
- $\tau \sim \xi^z$
- z dynamic scaling exponent.
- Collapse of the whole correlation function.
- $C(k,t) \equiv \langle \tilde{\phi}(k,0)\tilde{\phi}(-k,t)\rangle$
  where ϕ is the order parameter.

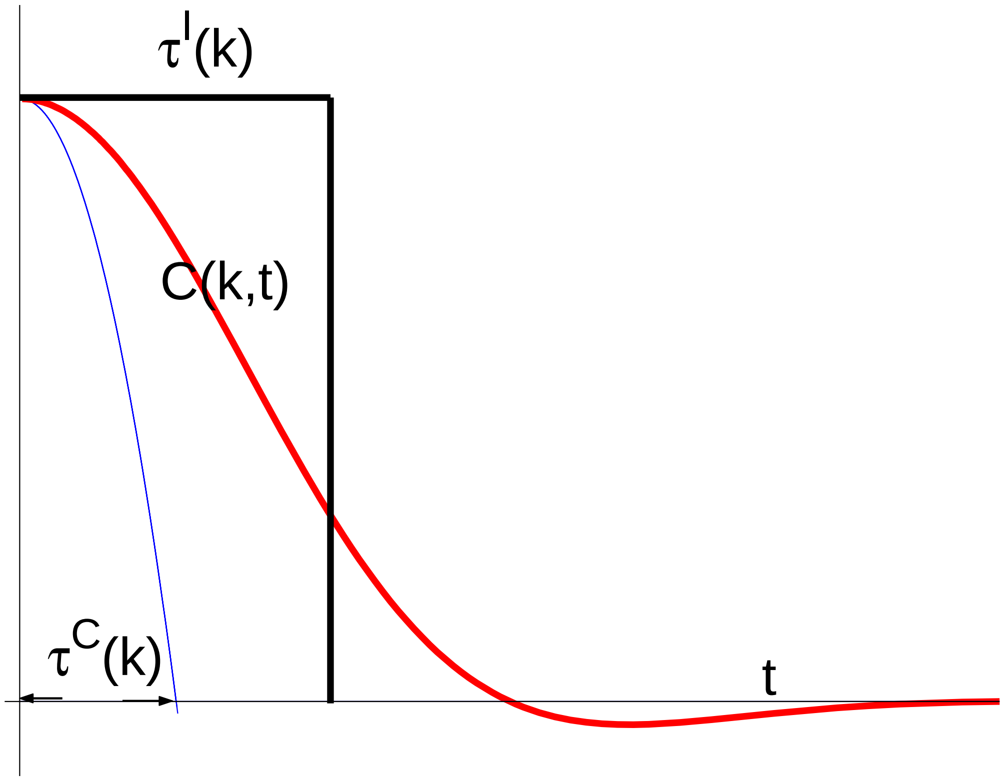
Data-Collapse (Schematic)



Each of the above curves in the first plot can be written in the form $f(t/\tau)$ where $\tau \sim k^{-z}$. Hence we obtain a collapse if we plot on the horizontal axis tk^z (second plot) instead of t itself.

Simple dynamic scaling

$$\tau^I(k) \sim \tau^C(k) \sim k^{-z}$$



Dynamic Structure Functions

$$\mathcal{F}_p(\ell, \{t_1, \ldots, t_{p-1}\}) \equiv \langle [\delta u_{\|}(\vec{x}, 0, \ell) \ldots \delta u_{\|}(\vec{x}, t_{p-1}, \ell)] \rangle.$$

Clearly $\mathcal{F}_p(\ell, \{t_1 = \ldots = t_{p-1} = 0) = \mathcal{S}_p(\ell)$. We normally restrict ourselves to the simple case $t_1 = t_2 = \ldots = t_q \equiv t$

Dynamic scaling

$$T_p^I(\ell) \equiv \frac{1}{S_p(\ell} \int_0^\infty \mathcal{F}(\ell, t) dt$$

$$\sim \ell^{z_p^I}$$

$$T_p^C(\ell) \equiv \left[\frac{1}{S_p(\ell)} \frac{\partial^2}{\partial t^2} \mathcal{F}(\ell, t)\right]^{-1/2}$$

$$\sim \ell^{z_p^C}$$

Dynamic multiscaling

- z_p depends on exactly how $\mathcal{T}_p(\ell)$ is extracted.
- No collapse of full correlation function.
  Breakdown of simple dynamic scaling.
- z_p is a non-linear function of p.

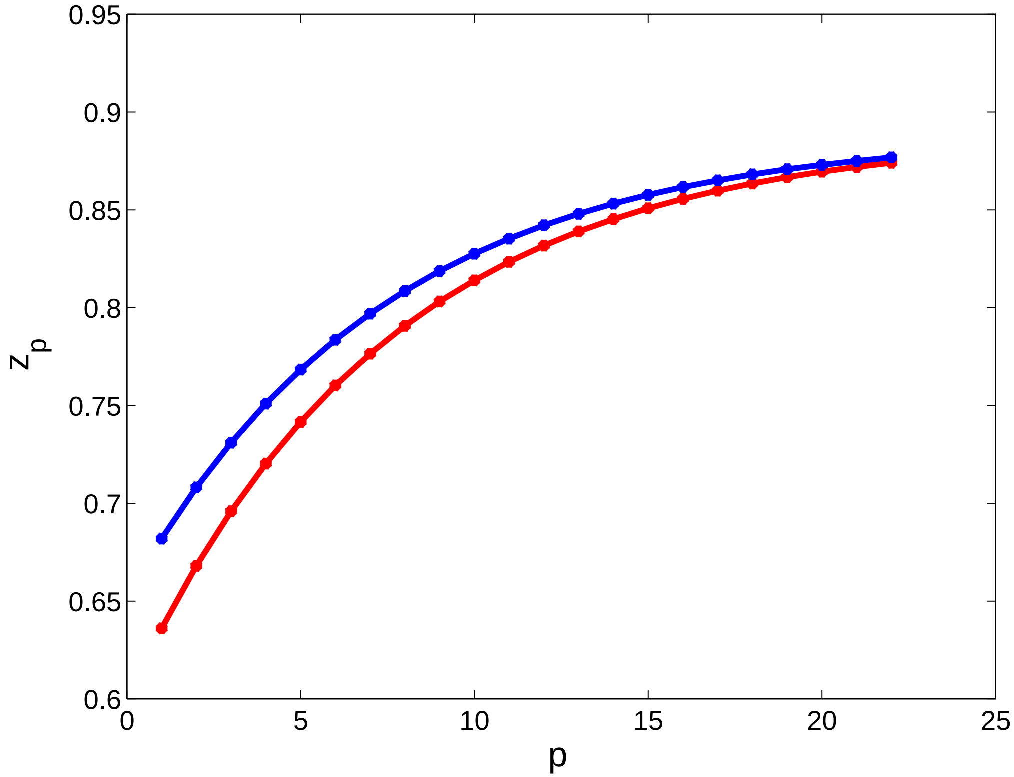
Summary of results

- Multifractal model predicts dynamic multiscaling

$$z^I_{p,1} = 1 + [\zeta_{p-1} - \zeta_p],$$

$$z^D_{p,2} = 1 + [\zeta_p - \zeta_{p+2}]/2.$$

Comparing Bridge Relations



The two bridge relations, using the SL formula for ζ_p, for $p = 1 \cdots 25$.[red (integral); blue (curvature)]

Summary of Results

- Fluid turbulence shows dynamic multiscaling.
- Breakdown of simple dynamic scaling
- Confirmed by numerical simulation in GOY shell model of turbulence.
- Can be generalised.
- No real analytical calculation is possible at present.
- If $\zeta_p = p/3$ (K41), simple dynamic scaling is obtained, $z_p = 2/3$.

Questions

1. Can dynamic scaling exponents be analytically extracted for other similar model ?
2. If answer to 1 is yes, then does multifractal model predicts correctly for 1.
3. Does equal-time multiscaling implies dynamic multiscaling ?
4. In general what are the ingredients of dynamic multiscaling ?

Answers

- Analytically tractable model exists, (Kraichnan model) but shows simple dynamic scaling,i.e. equal-time multiscaling does not necessarily imply dynamic multiscaling.
- Multifractal model gives correct result.
- If the advecting velocity is multiscaling, dynamic multiscaling is obtained.
- z_p need not necessary be a non-linear function of p for dynamic multiscaling.
- 3-d passive-vector should have same dynamic scaling like passive scalar.

Kraichnan Model

$$
\begin{aligned}
\frac{\partial}{\partial t}\theta + \vec{u}\cdot\nabla\theta &= \kappa\nabla^2\theta + f_\theta \\
\langle u_i(\vec{x},t)u_j(\vec{x}+\vec{\ell},t')\rangle &= 2D_{ij}(\vec{\ell})\delta(t-t') \\
D_{ij}(\vec{\ell}) &= D_o\delta_{ij} - \frac{1}{2}d_{ij}(\vec{\ell})
\end{aligned}
$$

$L \to \infty$ and $\eta \to 0$

$$
d_{ij} = D_1 \ell^{\xi}\left((d-1+\xi)\delta_{ij} - \xi\frac{\ell_i\ell_j}{\ell^2}\right).
$$

Equal-time multiscaling

- Multiscaling can be analytically (but perturbative) demonstrated.
- $S_p(\ell) \sim \ell^{\zeta_p^\theta}$.
- $0 < \xi < 2$.
- Structure functions have good limits, not correlation functions.
- 2-nd order quantities show dimensional scaling.
- Numerical simulations support analytical results.

Kraichnan shell model

$$[\frac{d}{dt} + \kappa k_m^2]\theta_m(t) = i[a_m(\theta^*_{m+1}u^*_{m-1} - \theta^*_{m-1}u^*_{m+1}) + b_m(\theta^*_{m-1}u^*_{m-2} + \theta^*_{m-2}u_{m-1}) + c_m(\theta^*_{m+2}u_{m+1} + \theta^*_{m+1}u^*_{m+2})] + \delta_{m,1}f(t).$$

$$\langle u_m(t)u_p^*(t')\rangle = D_m\delta_{p,m}\delta(t-t')$$

with $D_m = k_m^{-\xi}$. $S_p^\theta(m) \equiv \langle[\theta_m\theta_m^*]^{p/2}\rangle \sim k_m^{-\zeta_p^\theta}$

Dynamic structure functions

$$F_p(m,t) = \langle [\theta_m(0)\theta_m^*(t)]^{p/2} \rangle,$$

$F_2(m,t)$

$$\frac{d}{dt}F_2(m,t) = \langle \theta_m(0)[-\kappa k_m^2 \theta_m + \delta_{m,1} f]\rangle$$
$$- \langle \theta_m(0)\{i[a_m(\theta^*_{m+1}u^*_{m-1} - \theta^*_{m-1}u^*_{m+1})$$
$$+ b_m(\theta^*_{m-1}u^*_{m-2} + \theta^*_{m-2}u_{m-1})$$
$$+ c_m(\theta^*_{m+2}u_{m+1} + \theta^*_{m+1}u^*_{m+2})]\}\rangle,$$

$$\langle\theta_m(0)\theta^*_{m+1}u^*_{m-1}\rangle = \langle u^*_{m-1}u_q\rangle\langle\frac{\delta}{\delta u_q(t)}\theta_m(0)\theta^*_{m+1}(t)\rangle$$

(-15)

- $\langle u_m(t)u^*_p(t')\rangle = D_m\delta_{p,m}\delta(t-t')$
- Causality

Exponential decay in time

$$F_2(m,t) = S_2(m)e^{-t/\tau(m)},$$
$$\tau(m) = 4k_m^{-(2-\xi)}A^{-1}(\xi)$$

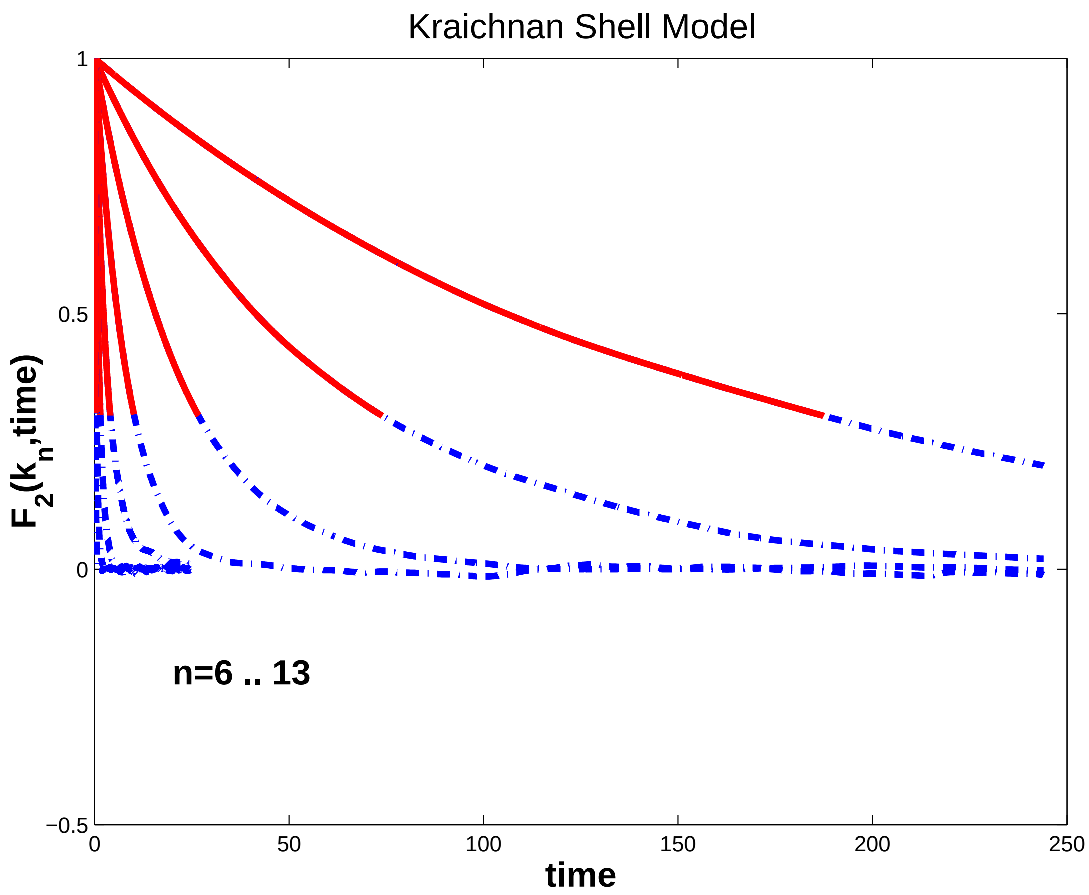
$$A(\xi) = [2^{(2\xi-2)} + 2^{-(2\xi-2)}] + [2^{\xi} + 2^{-\xi}] + [2^{(\xi-2)} + 2^{-(\xi-2)}].$$

- Similar relations can be obtained for higher values of p but the process is more cumbersome.

Numerical Simulation

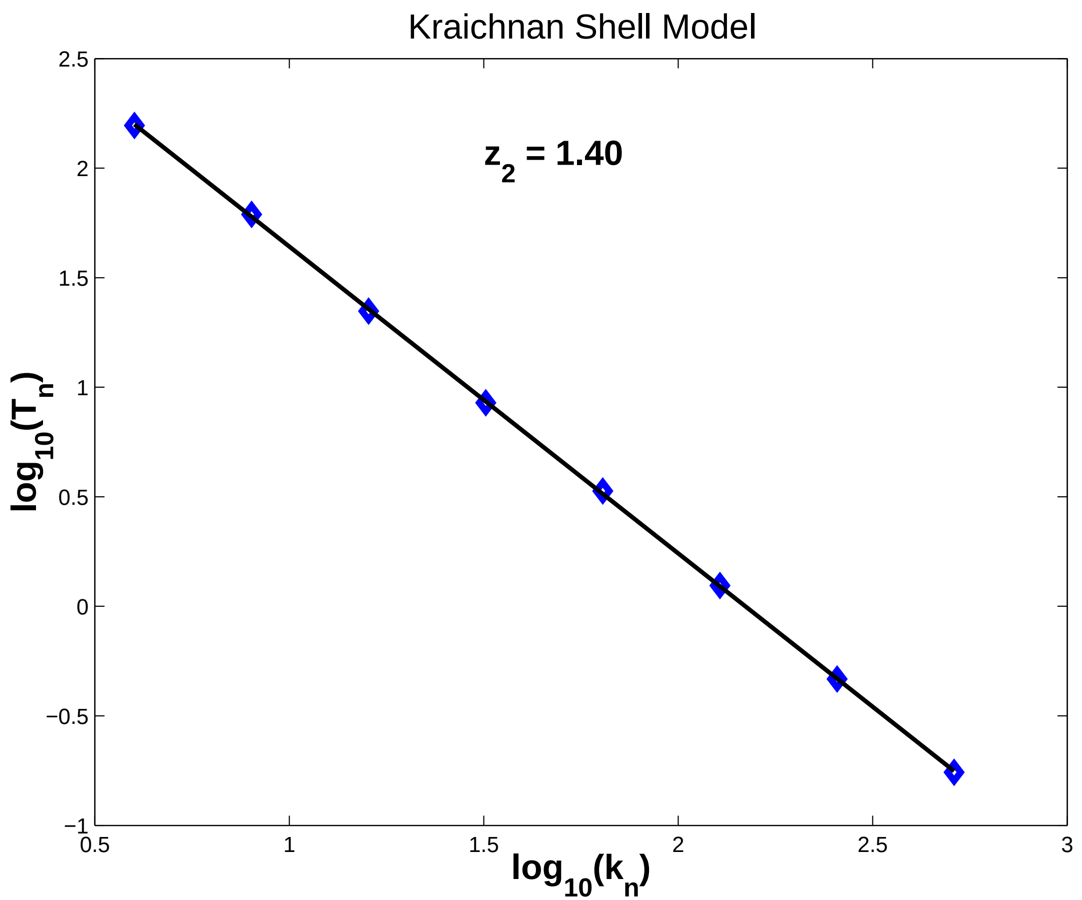
- Viscous term integrated exactly.
- Careful numerical scheme for the white-in-time velocity.

$F_2(m,t)(\xi = 0.6)$

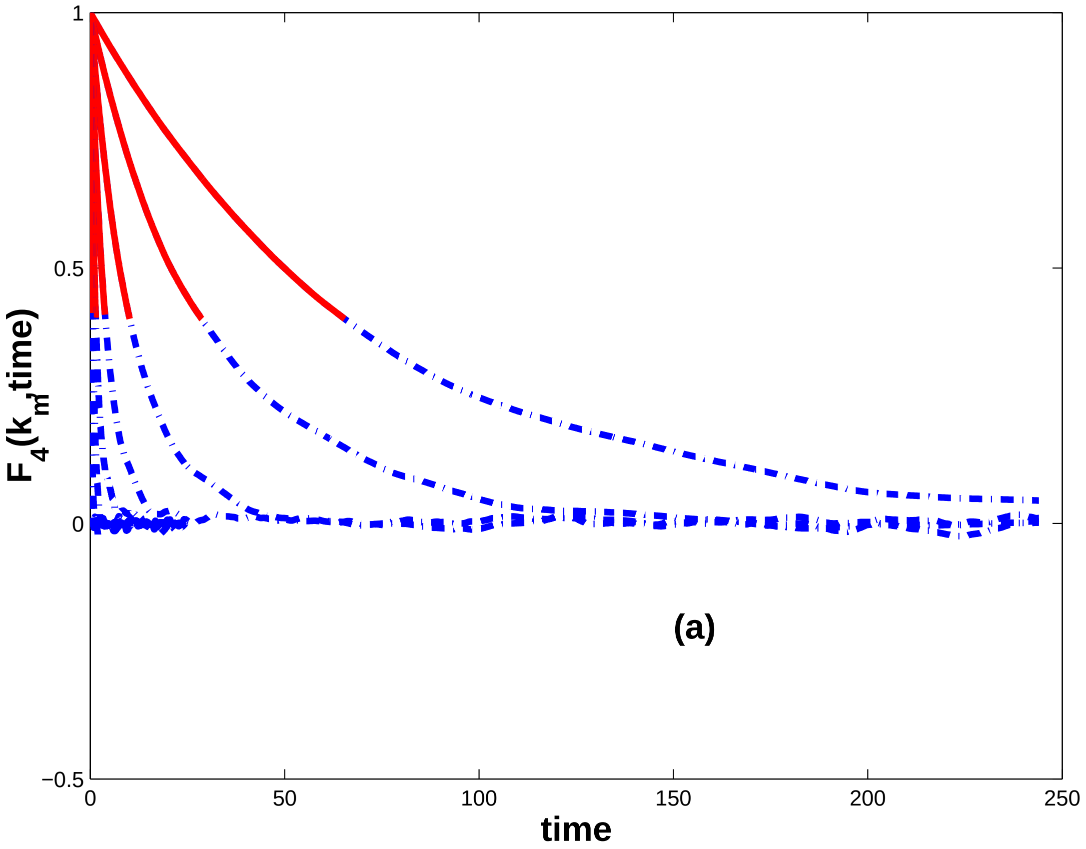


Kraichnan Shell Model
1
0.5
0
−0.5
0
50
100
150
200
250
time
F2(kn,time)
n=6 .. 13

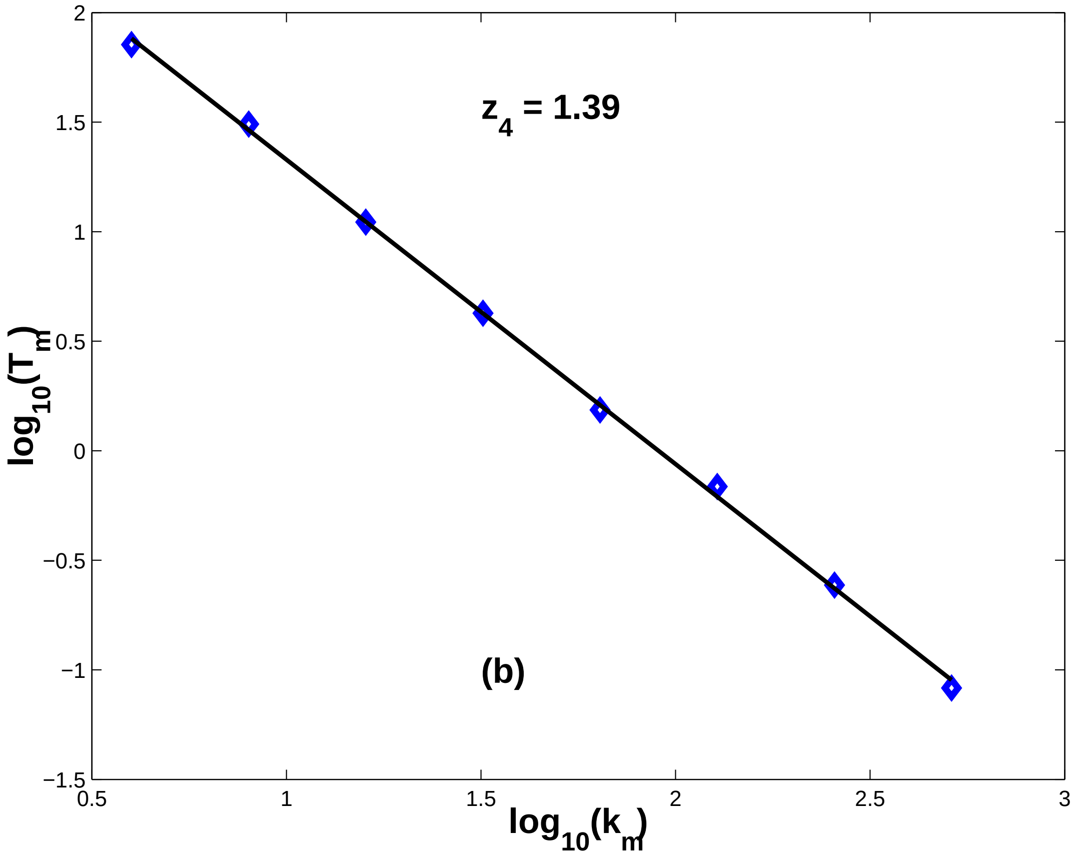
$z_2(\xi = 0.6)$


Kraichnan Shell Model
z2 = 1.40
log10(Tn)
log10(kn)
2.5
2
1.5
1
0.5
0
−0.5
−1
0.5
1
1.5
2
2.5
3

$F_4(m,t)(\xi = 0.6)$



$z_4(\xi = 0.6)$



Summary

- $z = 2 - \xi$
- Collapse for full correlation function.
- White-in-time nature of velocity plays major role.
- Equal-time multiscaling does not necessary imply dynamic multiscaling.
- Agrees with multifractal model predictions.
- Same analytical calculation applies to the full 3-d Kraichnan model.

More realistic velocity field

1. For velocity field with simple scaling but not white-in-time $z = 2 - \xi$ (multifractal model).
2. 1 should be true for passive-vector too. (Kinametic dynamo model)
3. What happens if velocity field is multiscaling ?

GOY Shell Model

$$[\frac{d}{dt} + \nu k_n^2]u_n =$$
$$i(a_n u_{n+1} u_{n+2} + b_n u_{n-1} u_{n+1} + c_n u_{n-1} u_{n-2})^* + f_n;$$

- dynamical variables: complex, scalar velocities u_n, for the shells n.
- one-dimensional, logarithmically spaced wavevectors k_n, i.e., $k_n = k_0 2^n$, and c.c is denoted by $*$
- $a_n = k_n$, $b_n = -\delta k_{n-1}$, $c_n = -(1-\delta)k_{n-2}$; chosen to conserve the shell-model analogues of energy and helicity in the

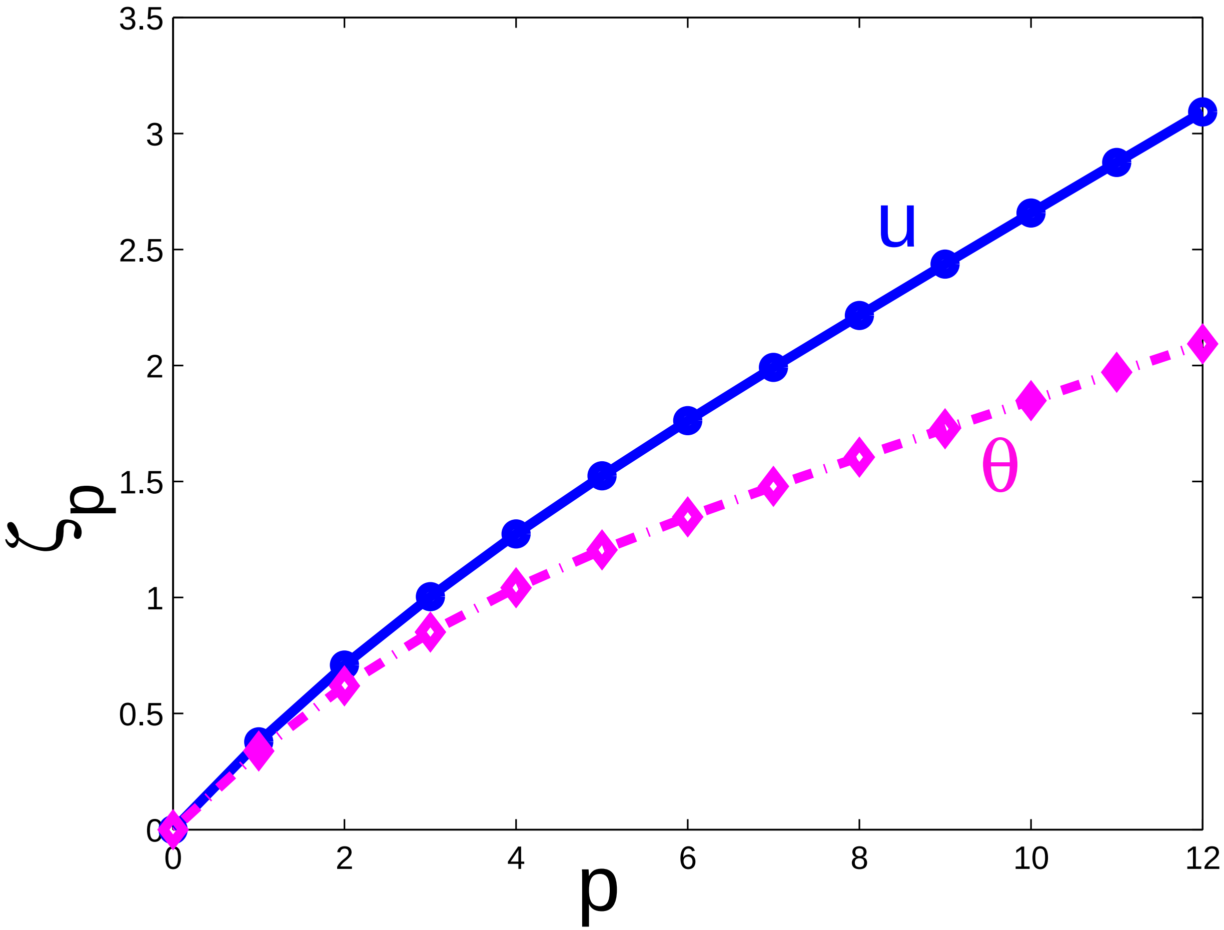
Passive scalar shell model

$$
\begin{aligned}
[\frac{d}{dt} + \kappa k_m^2]\theta_m(t) = i[a_m(\theta^*_{m+1}u^*_{m-1} - \theta^*_{m-1}u^*_{m+1}) \\
+ b_m(\theta^*_{m-1}u^*_{m-2} + \theta^*_{m-2}u^*_{m-1}) \\
+ c_m(\theta^*_{m+2}u^*_{m+1} + \theta^*_{m+1}u^*_{m+2})] \\
+ \delta_{m,1} f(t),
\end{aligned} \tag{-18}
$$

Equal-time properties

- Equal-time multiscaling.
- More intermittent than fluid.
- Agrees well with experiments.

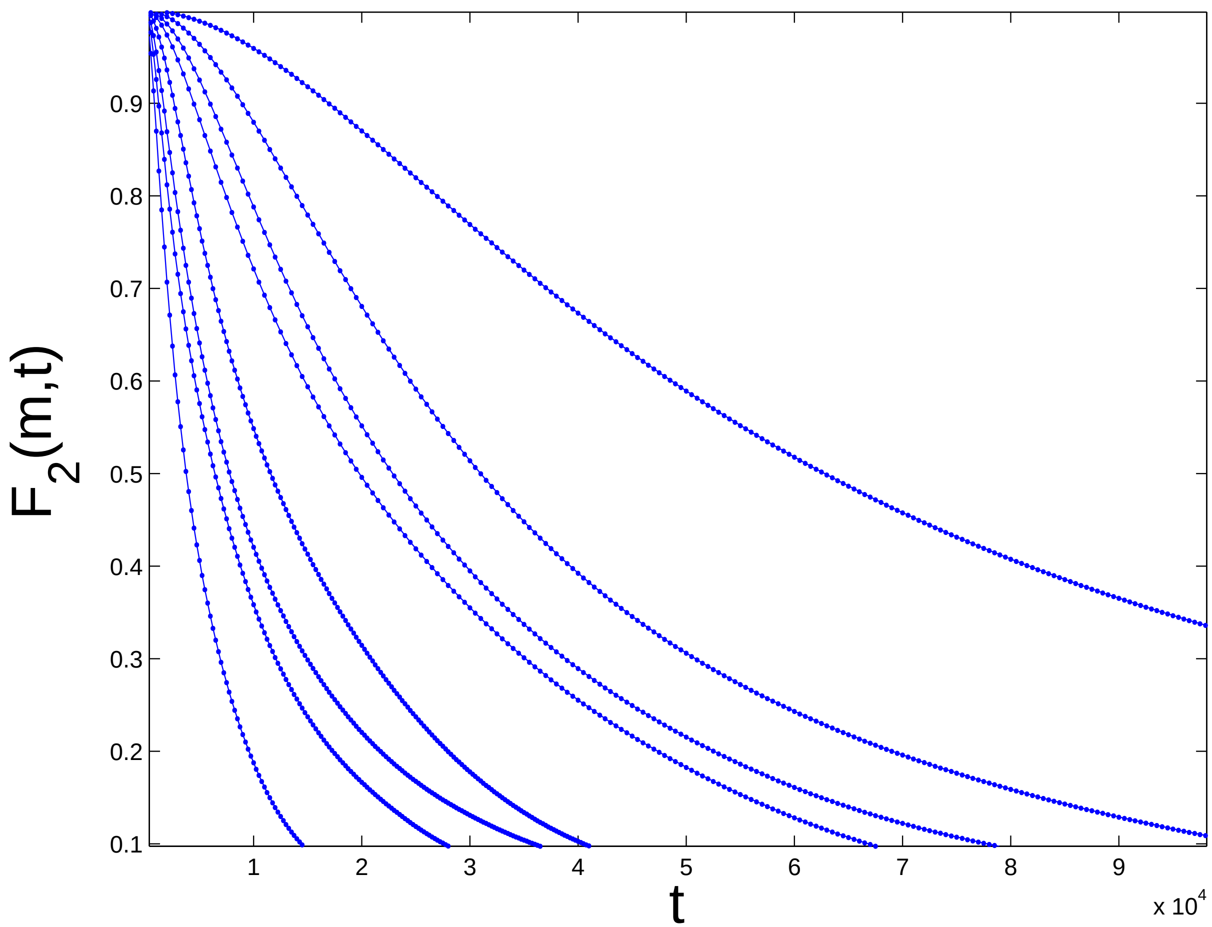
ζ_p^u and ζ_p^θ



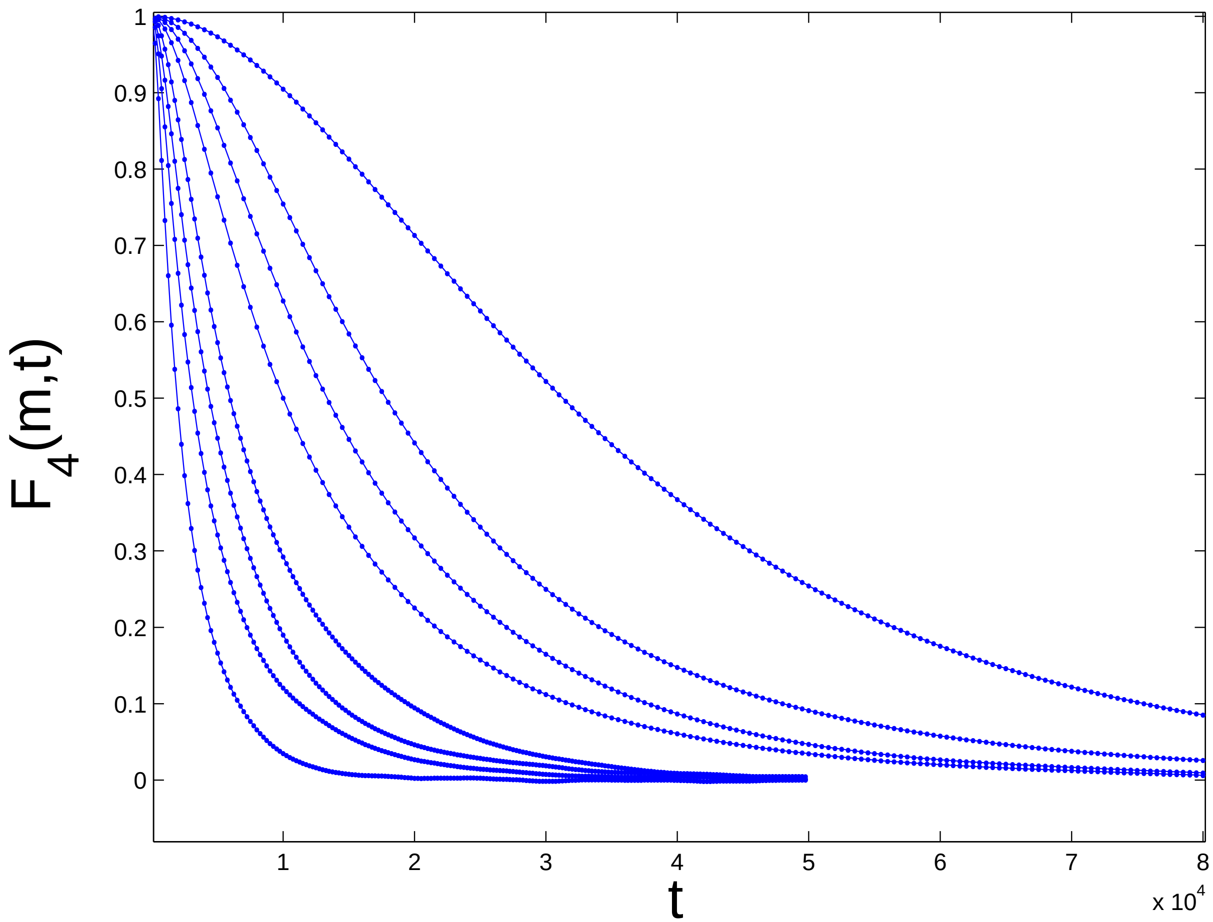
Time-series of u

Time-series of θ

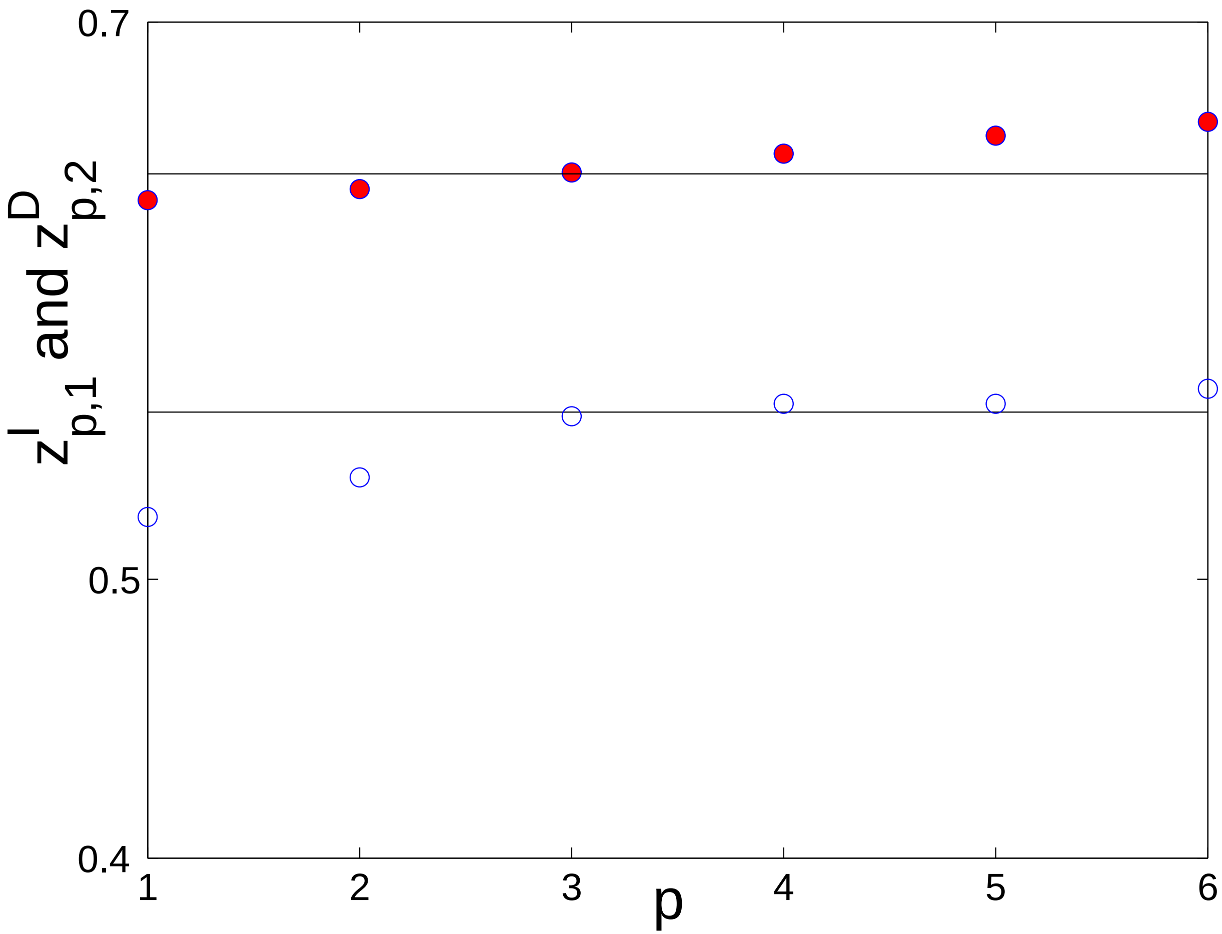
$F_2(m,t)$



$F_4(m, t)$



$z^D_{p,2}$

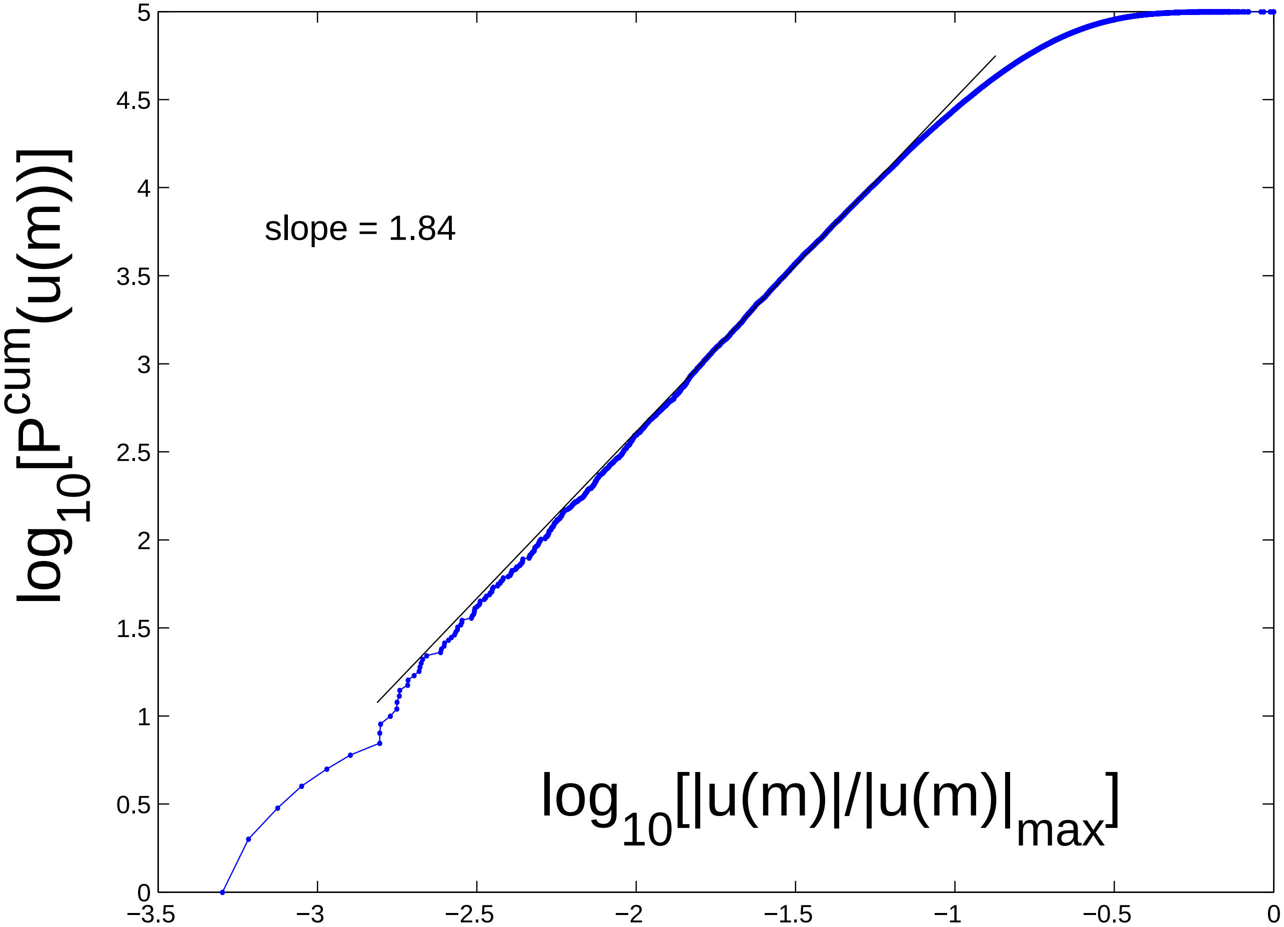


Dynamic Multiscaling

Multifractal model predicts:

- $z^D_{p,2} = 1 - \zeta^u_2/2$
- $z^I_{p,1} = 1 - \zeta^u_{-1}$
- Breakdown of simple scaling.
- Does structure functions with negative exponents exists ?

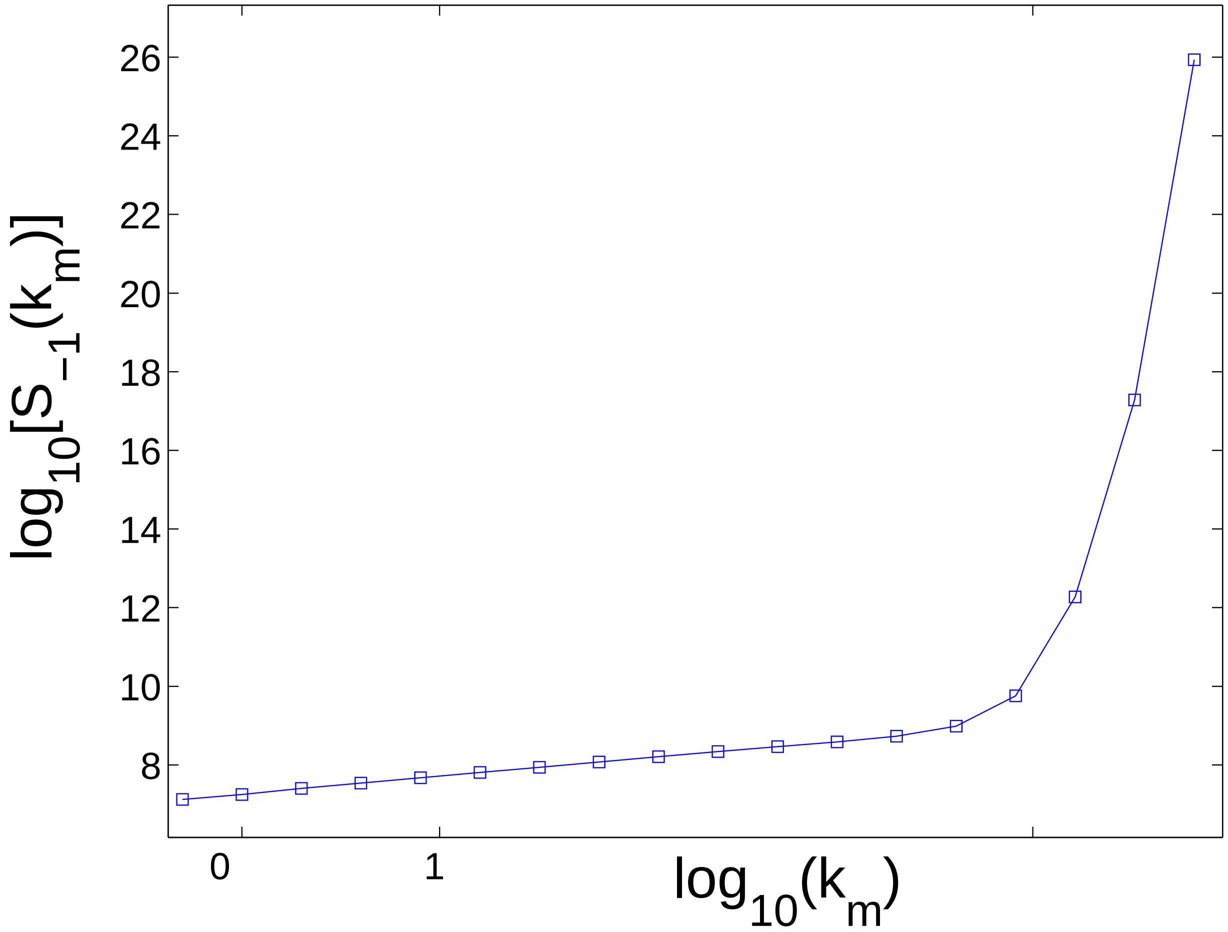
Cumuliative pdf for u_m



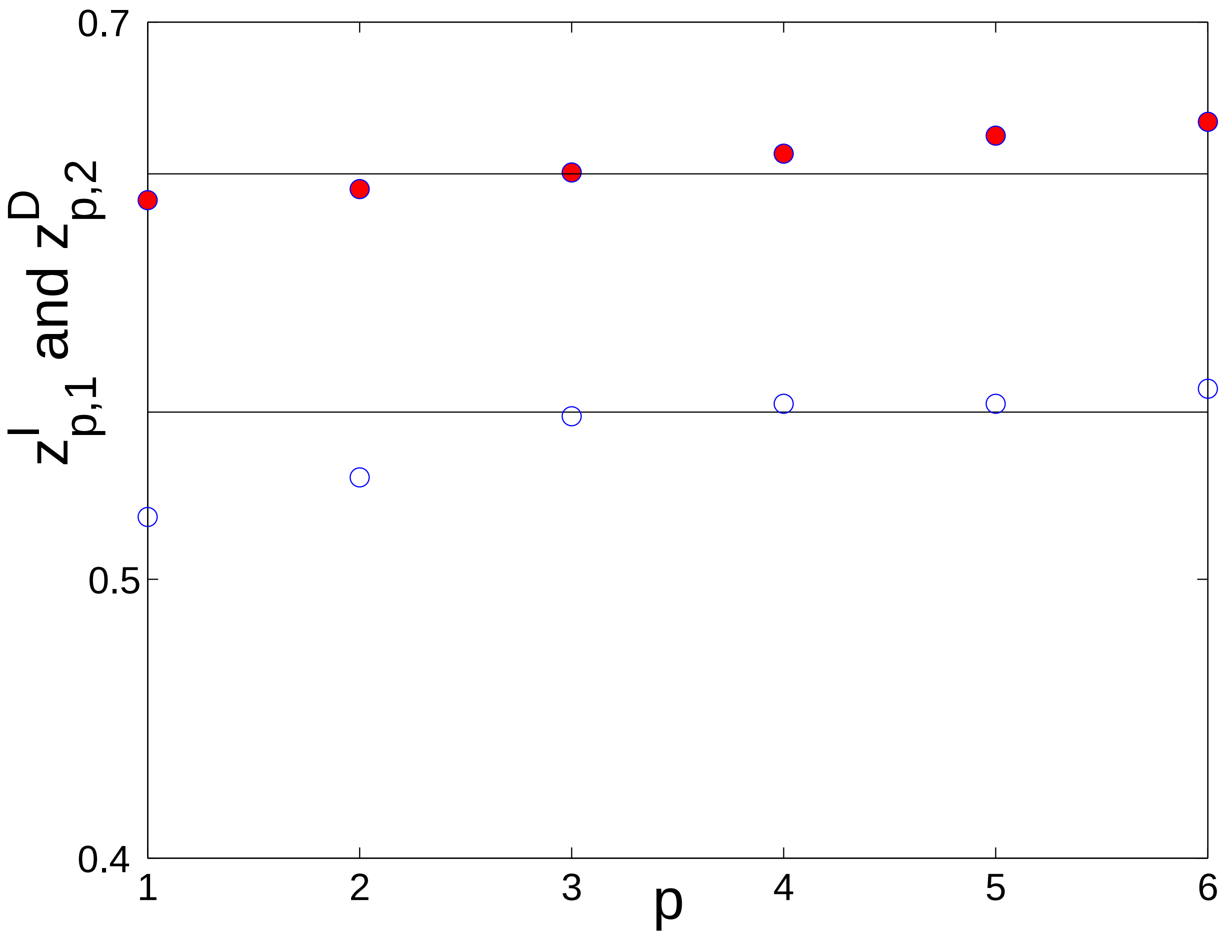
Negative exponents

- For small $|u_m|$, $P^{cum}[|u_m|] \sim |u_m|^{1.8}$.
- $P[|u_m|] \sim |u_m|^{0.8}$
- $S_{-1}(m) \equiv \int P[x]\frac{1}{x}dx \sim \int x^{-0.2}dx$; exists.
- But $S_p(m)$ for p, for $p \approx -1.8$ doesnot.
- $T^I_{p,M}$ for $M > 2$ doesnot exists.
- Measurement of a static quantity ($P(x)$) gives us information about existence of a dynamic quantity $T^I_{p,M}$.

ζ^u_{-1}



Dynamic multiscaling



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References

1. Dynamic Multiscaling in fluid turbulence : An Overview. **D. Mitra** and R. Pandit, *Physica A*, **318**, 179 (2003).
2. Varieties of Dynamic Multiscaling in Fluid Turbulence. **D. Mitra** and R. Pandit. *Phys. Rev. Lett.* **93**, 024501 (2004) (nlin.CD/0309037)
3. Dynamic scaling and multiscaling in passive scalar turbulence. **D. Mitra** and R. Pandit. Manuscript in preparation.